Canonical Kähler-Einstein metrics applied to a rigidity problem for irreducible lattices of rank ≥ 2 of the Hermitian type

Ngaiming Mok

Abstract: On a bounded Euclidean domain $D \Subset \mathbb{C}^n$, the most elementary canonical Kähler metric is the Bergman metric ds^2_D. By the invariance of ds^2_D under $\mathrm{Aut}(D)$, ds^2_D descends to any quotient of D by a torsion-free discrete subgroup $\Gamma' \subset \mathrm{Aut}(D)$ and can hence be used to study (quasi-)projective manifolds N uniformized by D. However, the Bergman metric ds^2_D and hence the induced Kähler metric ds^2_N are not necessarily complete, which limits the applicability of Kähler geometry. In contrast it was established by Cheng-Yau and Mok-Yau that a bounded domain of holomorphy admits up a unique canonical Kähler-Einstein metric of negative Ricci curvature -1. The same applies to bounded domains of holomorphy on Stein manifolds. In this talk we will show that such complete Kähler-Einstein metrics are applicable to study rigidity problems for holomorphic maps, when the domain manifold is an irreducible Shimura variety of rank ≥ 2 and the target is the quotient D/Γ' of a bounded domain of holomorphy $D \Subset Z$ on a Stein manifold Z by a torsion-free discrete subgroup $\Gamma' \subset \mathrm{Aut}(D)$ such that $N = D/\Gamma'$ is of bounded volume with respect to the Kobayashi-Royden volume form $d\mu$, the biggest canonical volume form enjoying the monotonicty property.

Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, and write $X_\Gamma := \Omega/\Gamma$. Let Z be a Stein manifold and $D \Subset Z$ be any bounded domain, $\Gamma' \subset \mathrm{Aut}(D)$ be a discrete subgroup such that $\mathrm{Volume}(Y_{\Gamma'}, d\mu) < \infty$. Let $F : \Omega \to D \Subset \mathbb{C}^N$ be a holomorphic map Γ-equivariant with respect to a group homomorphism $\Phi : \Gamma \to \Gamma'$. In a joint work with Kwok-Kin Wong, we prove that $F : \Omega \to D$ must be a biholomorphic map provided that $\Phi : \Gamma \to \Gamma'$ is a group isomorphism. We call this the Isomorphism Theorem.

To prove that F is a biholomorphism it suffices to be able to invert the holomorphic map. To do this we first construct a holomorphic map $R : D \to \Omega$ such that $R \circ F = \mathrm{id}_\Omega$. Hence, $F : \Omega \xrightarrow{\cong} F(\Omega)$ such that, writing $\varphi : F(\Omega) \to \Omega$ for its inverse, we have $R = \varphi \circ \varpi$ for a holomorphic retraction $\varpi : D \to F(\Omega)$. To construct R we introduce an averaging process on bounded holomorphic functions on Ω belonging to $\mathbf{H} := F^*H^\infty(D)$ in order to prove that there exist $h_1, \cdots, h_n \in H^\infty(D)$ such that $(F^*h_1, \cdots, F^*h_n) = \mathrm{id}_\Omega$. The averaging process involves harmonic analysis applied to certain complex submanifolds of Ω which are holomorphically and isometrically embedded copies of the complex unit ball of maximal dimension, and also Moore's ergodicity theorem on semisimple Lie groups. Finally, to prove that F is a biholomorphism it remains to show that the fibers of $\varpi : D \to F(\Omega)$ are 0-dimensional. When D is a domain of holomorphy we prove this by exploiting the geometry of $Y_{\Gamma'} = D/\Gamma'$ as a complete Kähler-Einstein manifold of finite volume. In general, we replace D by its hull of holomorphy $\widehat{D}$ and prove the same by deducing from the hypothesis $\mathrm{Volume}(Y_{\Gamma'}, d\mu) < \infty$ that $D \subset \widehat{D}$ is a schlicht domain such that $\widehat{D} - D$ is of zero Lebesgue measure.