全国大学生
数学竞赛
提高课程

01

第八章 多元函数微分学

中国大学MOOC



内容

01 偏导数的基本概念

02 多元复合函数求导法则

03 多元隐函数求导法则

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一、偏导数的基本概念

定义（偏导数） 设二元函数$f(x,y)$在点$P_0(x_0,y_0)$的某邻域内有定义，当$y=y_0$固定不变时，一元函数$f(x,y_0)$在$x=x_0$处可导，即极限

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}$$

存在，则称此极限值为函数$z=f(x,y)$在点$P_0(x_0,y_0)$处关于x的偏导数.记为：$f_x'(x_0,y_0)$，$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}$，$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$，$z_x'(x_0,y_0)$

$$\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

一、偏导数的基本概念

当$z=f(x,y)$在区域D内偏导数存在时，偏导数$f_x'(x,y)$和$f_y'(x,y)$仍为x,y的函数，所以也称$f_x'(x,y)$和$f_y'(x,y)$为偏导函数.

求二元函数$z=f(x,y)$偏导数的方法：

(1) 把y视为常数，用一元函数的求导法则求$\dfrac{\partial z}{\partial x}$；

(2) 把x视为常数，用一元函数的求导法则求$\dfrac{\partial z}{\partial y}$.

一、偏导数的基本概念

对于三元函数 $u = f(x, y, z)$ 在点 (x, y, z) 处对自变量的偏导数：

$$f'_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$f'_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$f'_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

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例　设可微函数$f(x,y)$满足$\frac{\partial f}{\partial x}=-f(x,\ y)$，$f(0,\frac{\pi}{2})=1$，且$\lim\limits_{n\to\infty}\left|\frac{f(0,\ y+\frac{1}{n})}{f(0,\ y)}\right|^{n}=e^{\cot y}$，则$f(x,y)=$_____.

解　由$\lim\limits_{n\to\infty}\left|\frac{f(0,\ y+\frac{1}{n})}{f(0,\ y)}\right|^{n}=e^{\cot y}$，等式两端取对数得，

$$\cot y=\lim_{n\to\infty}\frac{\ln f(0,\ y+\frac{1}{n})-\ln f(0,\ y)}{\frac{1}{n}}=\frac{\partial}{\partial y}[\ln f(0,\ y)]=\frac{f_y'(0,\ y)}{f(0,\ y)}$$

一、偏导数的基本概念

设可微函数 $f(x,y)$ 满足 $\dfrac{\partial f}{\partial x}=-f(x,y)$，$f(0,\dfrac{\pi}{2})=1$，且 $\lim\limits_{n\to\infty}\left[\dfrac{f(0,\ y+\frac{1}{n})}{f(0,\ y)}\right]^n=e^{\cot y}$，

则 $f(x,y)=$_____.

$\cot y=\dfrac{f_y'(0,\ y)}{f(0,\ y)}$　所以　$\dfrac{f_y'(0,\ y)}{f(0,\ y)}=\dfrac{(\sin y)'}{\sin y}$，

于是得　$\ln f(0,\ y)=\ln\sin y+\ln C$，即 $f(0,\ y)=C\sin y$

从而 $f(x,y)=g(x)\sin y$

这时　$\dfrac{\partial f}{\partial x}=g'(x)\sin y=-f(x,y)=-g(x)\sin y$，

一、偏导数的基本概念

设可微函数$f(x,y)$满足$\frac{\partial f}{\partial x}=-f(x,y)$，$f(0,\frac{\pi}{2})=1$，且$\lim\limits_{n\to\infty}\left[\frac{f(0,y+\frac{1}{n})}{f(0,y)}\right]^n=e^{\cot y}$，则$f(x,y)=$_____.

由y的任意性得，$g'(x)=-g(x)$，即$\ln g(x)=-x+\ln C_1$，$g(x)=C_1e^{-x}$.

$f(x,y)=g(x)\sin y$

由$f(0,\frac{\pi}{2})=1$得 $1=g(0)\sin\frac{\pi}{2}=C_1$，故$f(x,y)=e^{-x}\sin y$

二、多元复合函数求导法则

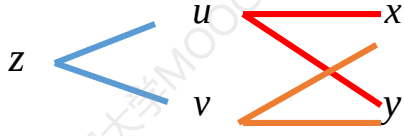
定义　设函数$z=f(u,v)$是变量u,v的函数，而u,v又是变量x,y的函数，$u=\varphi(x,y)$、$v=\psi(x,y)$，因此$z=f[\varphi(x,y),\psi(x,y)]$是$x,y$的复合函数.

基本形式的复合函数偏导数链式法则

设函数$u=\varphi(x,y),v=\psi(x,y)$在点$(x,y)$处偏导数存在，且在对应$(x,y)$的点$(u,v)$处，函数$z=f(u,v)$可微，则复合函数$f[\varphi(x,y),\psi(x,y)]$在点$(x,y)$处两个偏导数均存在，且

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$



二、多元复合函数求导法则

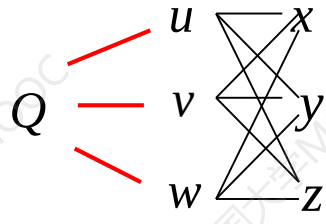
三元函数 $Q = f(u, v, w)$，

其中 $u = \varphi(x, y, z)$、$v = \psi(x, y, z)$、$w = \omega(x, y, z)$

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial Q}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial Q}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial Q}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial Q}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial Q}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial Q}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial Q}{\partial w} \cdot \frac{\partial w}{\partial z}$$

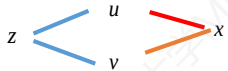


其它形式复合函数偏导数的链式法则

(1) 若 $z=f(u,v)$ 具有连续偏导数，而 $u=\varphi(x), v=\psi(x)$ 可导，则复合函数 $z=f[\varphi(x),\psi(x)]$ 有

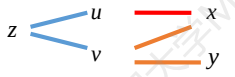
$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}$$

($\frac{\mathrm{d}z}{\mathrm{d}x}$ 称为 z 对 x 的全导数).



(2) 若 $u=\varphi(x)$ 在点 x 处可导，$v=\psi(x,y)$ 有连续偏导数，函数 $z=f(u,v)$ 在对应点 (u,v) 具有连续偏导数，则复合函数 $z=f[\varphi(x),\psi(x,y)]$ 在 (x,y) 处的两个偏导数存在，且

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

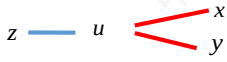


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二、多元复合函数求导法则

(3) 若 $z = f(u)$ 可导，$u = \varphi(x, y)$ 有连续偏导数，则复合函数 $z = f[\varphi(x, y)]$ 偏导数存在，且有

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}u} \cdot \frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\mathrm{d}z}{\mathrm{d}u} \cdot \frac{\partial u}{\partial y}$$



二、多元复合函数求导法则

例1　设 $w = f(u,v)$ 具有二阶连续偏导数，且 $u = x - cy$, $v = x + cy$，其中 c 为非零常数，则 $w_{xx} - \frac{1}{c^2} w_{yy} =$ _____.

解　$w_x = f_1 + f_2$,　$w_{xx} = (f_{11} + f_{12}) + (f_{21} + f_{22}) = f_{11} + 2f_{12} + f_{22}$

$w_y = f_1 \cdot (-c) + f_2 \cdot c = c(-f_1 + f_2)$,

$w_{yy} = c \cdot \frac{\partial}{\partial y}(-f_1 + f_2) = c\left[-f_{11} \cdot (-c) - f_{12} \cdot c + f_{21} \cdot (-c) + f_{22} \cdot c\right]$

$= c\left(cf_{11} - cf_{12} - cf_{21} + cf_{22}\right) = c^2\left(f_{11} - 2f_{12} + f_{22}\right)$

故 $w_{xx} - \frac{1}{c^2} w_{yy} = f_{11} + 2f_{12} + f_{22} - \frac{1}{c^2} \cdot c^2\left(f_{11} - 2f_{12} + f_{22}\right) = 4f_{12}$.

二、多元复合函数求导法则

例 2　已知 $z = xf\left(\frac{y}{x}\right) + 2y\varphi\left(\frac{x}{y}\right)$，其中 f，φ 均为二次可微函数.

(1) 求 $\frac{\partial z}{\partial x}$，$\frac{\partial^2 z}{\partial x \partial y}$；　(2) 当 $f = \varphi$，且 $\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{x=a} = -by^2$ 时，求 $f(y)$.

解　(1) $\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right) + x \cdot f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) + 2y \cdot \varphi'\left(\frac{x}{y}\right) \cdot \frac{1}{y} = f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right) + 2\varphi'\left(\frac{x}{y}\right)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left[f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right) + 2\varphi'\left(\frac{x}{y}\right)\right]$$

$$= f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} - \left[\frac{1}{x} f'\left(\frac{y}{x}\right) + \frac{y}{x} \cdot f''\left(\frac{y}{x}\right) \cdot \frac{1}{x}\right] + 2\varphi''\left(\frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right)$$

$$= f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} - \frac{1}{x} f'\left(\frac{y}{x}\right) - \frac{y}{x^2} f''\left(\frac{y}{x}\right) - \frac{2x}{y^2}\varphi''\left(\frac{x}{y}\right) = -\frac{y}{x^2} f''\left(\frac{y}{x}\right) - \frac{2x}{y^2}\varphi''\left(\frac{x}{y}\right).$$

二、多元复合函数求导法则

(2) 当 $f=\varphi$，且 $\left.\dfrac{\partial^2 z}{\partial x\partial y}\right|_{x=a}=-by^2$ 时，求 $f(y)$.

$$(2)\ \left.\frac{\partial^2 z}{\partial x\partial y}\right|_{x=a}=-\frac{y}{x^2}f''\left(\frac{y}{x}\right)-\frac{2x}{y^2}\varphi''\left(\frac{x}{y}\right)\Bigg|_{x=a}=-\frac{y}{a^2}f''\left(\frac{y}{a}\right)-\frac{2a}{y^2}\varphi''\left(\frac{a}{y}\right)=-by^2$$

因为 $f=\varphi$，所以 $\dfrac{y}{a^2}f''\left(\dfrac{y}{a}\right)+\dfrac{2a}{y^2}f''\left(\dfrac{a}{y}\right)=-by^2$，　令 $u=\dfrac{y}{a}$

则 $\dfrac{u}{a}f''(u)+\dfrac{2}{au^2}f''\left(\dfrac{1}{u}\right)=a^2bu^2$，　$u^3f''(u)+2f''\left(\dfrac{1}{u}\right)=a^3bu^4$　①

① 中用 $\dfrac{1}{u}$ 替换 u 得　$f''\left(\dfrac{1}{u}\right)+2u^3f''(u)=\dfrac{a^3b}{u}$　②

二、多元复合函数求导法则

$$\begin{cases} u^3 f''(u) + 2 f''\left(\dfrac{1}{u}\right) = a^3 b u^4 & ① \\ 2u^3 f''(u) + f''\left(\dfrac{1}{u}\right) = \dfrac{a^3 b}{u} & ② \end{cases}$$

(2) 当 $f = \varphi$，且 $\left.\dfrac{\partial^2 z}{\partial x \partial y}\right|_{x=a} = -by^2$ 时，求 $f(y)$.

令 $u = \dfrac{y}{a}$

$$-3u^3 f''(u) = a^3 b\left(u^4 - \frac{2}{u}\right) \Rightarrow f''(u) = \frac{a^3 b}{3}\left(\frac{2}{u^4} - u\right)$$

$$\Rightarrow f'(u) = \frac{a^3 b}{3}\left(-\frac{2}{3u^3} - \frac{1}{2}u^2\right) + C_1 \quad \Rightarrow f(u) = \frac{a^3 b}{3}\left(\frac{1}{3u^2} - \frac{1}{6}u^3\right) + C_1 u + C_2$$

把 $u = \dfrac{y}{a}$ 回代得 $f(y) = \dfrac{a^3 b}{3}\left(\dfrac{1}{3y^2} - \dfrac{1}{6}y^3\right) + C_1 y + C_2$.

三、多元隐函数求导法则

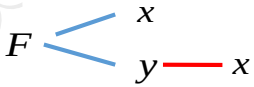
一个方程的情形

$$F(x,y)=0 \rightarrow y=f(x)$$

$$F(x,y,z)=0 \rightarrow z=f(x,y)$$

隐函数存在定理　设函数$F(x,y)$在点$P_0(x_0,y_0)$的某邻域具有连续的偏导数，且$F(x_0,y_0)=0$，$F_y'(x_0,y_0)\neq 0$，则方程$F(x,y)=0$在点P_0的某邻域内唯一确定一个具有连续导数的函数$y=f(x)$，它满足$y_0=f(x_0)$，且

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x'}{F_y'}.$$

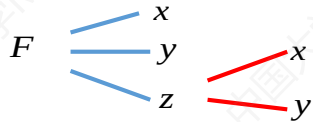


三、多元隐函数求导法则

类似地，对于由方程$F(x,y,z)=0$确定的二元隐函数$z=f(x,y)$，

若函数$F(x,y,z)$存在且$\dfrac{\partial F}{\partial z}\neq 0$，则有

$$F\left(x,y,f(x,y)\right)\equiv 0$$



两边对x求导 $\dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial z}\cdot\dfrac{\partial z}{\partial x}=0$，得 $\dfrac{\partial z}{\partial x}=-\dfrac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}=-\dfrac{F_x'}{F_z'}$

两边对y求导 $\dfrac{\partial F}{\partial y}+\dfrac{\partial F}{\partial z}\cdot\dfrac{\partial z}{\partial y}=0$，得 $\dfrac{\partial z}{\partial y}=-\dfrac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}=-\dfrac{F_y'}{F_z'}$

三、多元隐函数求导法则

例1　设函数$z=z(x,y)$由方程$F(x+\frac{z}{y},y+\frac{z}{x})=0$所决定，其中$F(u,v)$具有连续的偏导数，且$xF_u+yF_v\neq 0$，则$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$_____.(本小题结果要求不显含$F$及其偏导数)

解　方程两端关于x求偏导数可得　$(1+\frac{1}{y}\frac{\partial z}{\partial x})F_u+(-\frac{z}{x^2}+\frac{1}{x}\frac{\partial z}{\partial x})F_v=0,$

解得　$\frac{\partial z}{\partial x}=\frac{y(zF_v-x^2F_u)}{x(xF_u+yF_v)}.$　　对y求偏导解得　$\frac{\partial z}{\partial y}=\frac{x(zF_u-y^2F_v)}{y(xF_u+yF_v)},$

于是　$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{y(zF_v-x^2F_u)}{x(xF_u+yF_v)}+\frac{x(zF_u-y^2F_v)}{xF_u+yF_v}=\frac{z(xF_u+yF_v)-xy(xF_u+yF_v)}{xF_u+yF_v}$

$=z-xy.$

三、多元隐函数求导法则

例2　设函数$z=z(x,y)$是由方程$F(x-y,z)=0$确定，其中$F(u,v)$具有二阶连续偏导数，则$\dfrac{\partial^2 z}{\partial x\partial y}=$_____.

解　方程$F(x-y,z)=0$关于x求导得$F_1'+F_2'\dfrac{\partial z}{\partial x}=0$，解得$\dfrac{\partial z}{\partial x}=-\dfrac{F_1'}{F_2'}$，

方程$F(x-y,z)=0$关于y求导，得$-F_1'+F_2'\dfrac{\partial z}{\partial y}=0$，可得$\dfrac{\partial z}{\partial y}=\dfrac{F_1'}{F_2'}$.

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{F_1'}{F_2'}\right)=-\frac{\dfrac{\partial F_1'}{\partial y}\cdot F_2'-F_1'\cdot\dfrac{\partial F_2'}{\partial y}}{F_2'^2}$$

$$=-\frac{\left(-F_{11}''+F_{12}''\dfrac{\partial z}{\partial y}\right)F_2'-F_1'\left(-F_{21}''+F_{22}''\dfrac{\partial z}{\partial y}\right)}{F_2'^2}=\frac{F_1'^2F_{22}''-2F_1'F_2'F_{12}''+F_2'^2F_{11}''}{F_2'^3}$$



END

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