全国大学生
数学竞赛



第8届决赛

中国大学MOOC

一、填空题

(本题共5个小题，每小题6分，共30分)

第1题



中国大学MOOC

(2017决赛,6分)

过单叶双曲面$\frac{x^2}{4}+\frac{y^2}{2}-2z^2=1$与球面$x^2+y^2+z^2=4$的交线且与直线

$L:\begin{cases}x=0\\3y+z=0\end{cases}$垂直的平面方程为_____.

解 直线L的方向向量$\vec{s}=\{1,0,0\}\times\{0,3,1\}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&0&0\\0&3&1\end{vmatrix}=-\vec{j}+3\vec{k}.$

所求平面的法向向量为$\{0,1,-3\}$.

联立$\begin{cases}\frac{x^2}{4}+\frac{y^2}{2}-2z^2=1,\\x^2+y^2+z^2=4,\end{cases}$消去$x^2$可得$y^2=9z^2$,

两曲面交线上的点满足$y-3z=0$或$y+3z=0$,

过单叶双曲面$\frac{x^2}{4}+\frac{y^2}{2}-2z^2=1$与球面$x^2+y^2+z^2=4$的交线且与直线$\begin{cases}x=0\\3y+z=0\end{cases}$垂直的平面方程为_____.

两曲面交线上的点满足$y-3z=0$或$y+3z=0$,

也即过两曲面交线上的点的平面应该满足：$y-3z=0$或$y+3z=0$,

由所求平面的法向向量为$(0,1,-3)$得所求平面方程为$y-3z=0$.

一、填空题

(本题共5个小题，每小题6分，共30分)

第2题



中国大学MOOC

3.(2017决赛,6分) 设可微函数$f(x,y)$满足$\frac{\partial f}{\partial x}=-f(x,\ y)$，$f(0,\frac{\pi}{2})=1$，且$\lim\limits_{n\to\infty}\left(\frac{f(0,\ y+\frac{1}{n})}{f(0,\ y)}\right)^n=\mathrm{e}^{\cot y}$，则$f(x,y)=$_____.

解　由$\lim\limits_{n\to\infty}\left(\frac{f(0,\ y+\frac{1}{n})}{f(0,\ y)}\right)^n=\mathrm{e}^{\cot y}$，等式两端取对数得，

$$\cot y=\lim_{n\to\infty}\frac{\ln f(0,\ y+\frac{1}{n})-\ln f(0,\ y)}{\frac{1}{n}}=\frac{\partial}{\partial y}[\ln f(0,\ y)]=\frac{f_y'(0,\ y)}{f(0,\ y)}$$

(2017决赛,6分) 设可微函数$f(x,y)$满足$\frac{\partial f}{\partial x}=-f(x,\ y)$，$f(0,\frac{\pi}{2})=1$，且$\lim\limits_{n\to\infty}\left(\frac{f(0,\ y+\frac{1}{n})}{f(0,\ y)}\right)^n=e^{\cot y}$，则$f(x,y)=$_____.

$\cot y=\frac{f_y'(0,\ y)}{f(0,\ y)}$　　所以　$\frac{f_y'(0,\ y)}{f(0,\ y)}=\frac{(\sin y)'}{\sin y}$,

于是得　$\ln f(0,\ y)=\ln\sin y+C_1$，即$f(0,\ y)=C_2\sin y$

从而$f(x,y)=g(x)\sin y$

这时　$\frac{\partial f}{\partial x}=g'(x)\sin y=-f(x,y)=-g(x)\sin y,$

(2017决赛,6分) 设可微函数$f(x,y)$满足$\frac{\partial f}{\partial x}=-f(x,\ y)$，$f(0,\frac{\pi}{2})=1$，且$\lim\limits_{n\to\infty}\left(\frac{f(0,\ y+\frac{1}{n})}{f(0,\ y)}\right)^n=\mathrm{e}^{\cot y}$，则$f(x,y)=$_____.

由y的任意性得，$g'(x)=-g(x)$，即$\ln g(x)=-x+C_3$，$g(x)=C_4\mathrm{e}^{-x}$.

$f(x,y)=g(x)\sin y$

由$f(0,\frac{\pi}{2})=1$得 $1=g(0)\sin\frac{\pi}{2}=C_4$,

故$f(x,y)=\mathrm{e}^{-x}\sin y$

一、填空题

(本题共5个小题，每小题6分，共30分)

第3题



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(2017决赛,6分)

已知$\boldsymbol{A}$为n阶可逆反对称矩阵，$\boldsymbol{b}$为n元列向量，设$\boldsymbol{B}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{b}\\ \boldsymbol{b}^{\mathrm{T}} & 0\end{pmatrix}$，则$\mathrm{rank}(\boldsymbol{B})=$(　　).

解　由于$\boldsymbol{A}$为可逆反对称矩阵，所以$\boldsymbol{A}^{-1}$存在且$\boldsymbol{A}^{\mathrm{T}}=-\boldsymbol{A}$，而$\boldsymbol{b}$为列向量，

所以 $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}=(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b})^{\mathrm{T}}=\boldsymbol{b}^{\mathrm{T}}(\boldsymbol{A}^{-1})^{\mathrm{T}}\boldsymbol{b}=\boldsymbol{b}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{-1}\boldsymbol{b}=\boldsymbol{b}^{\mathrm{T}}(-\boldsymbol{A}^{-1})\boldsymbol{b}=-\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}$,

故$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}=0$.

对$\boldsymbol{B}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{b}\\ \boldsymbol{b}^{\mathrm{T}} & 0\end{pmatrix}$先后做列初等变换和行初等变换，可得

$$\boldsymbol{B}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{b}\\ \boldsymbol{b}^{\mathrm{T}} & 0\end{pmatrix}\to\begin{pmatrix}\boldsymbol{A} & \boldsymbol{b}-\boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{b}\\ \boldsymbol{b}^{\mathrm{T}} & -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}\end{pmatrix}\to\begin{pmatrix}\boldsymbol{A} & 0\\ \boldsymbol{b}^{\mathrm{T}}-\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b} & 0\end{pmatrix}\to\begin{pmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & 0\end{pmatrix}.$$

从而得$\mathrm{rank}(\boldsymbol{B})=\mathrm{rank}(\boldsymbol{A})=n$.

一、填空题

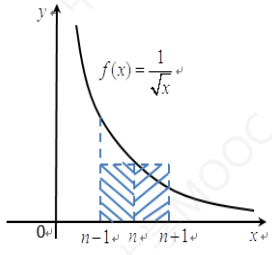
(本题共5个小题，每小题6分，共30分)

第4题



中国大学MOOC

(2017决赛,6分) $\sum_{n=1}^{100} n^{-\frac{1}{2}}$的整数部分为(　　).



解　设$f(x)=\dfrac{1}{\sqrt{x}}$，则$f(x)$在$(0,+\infty)$上非负且单调递减，

所以 $\int_n^{n+1} f(x)\mathrm{d}x < f(n) < \int_{n-1}^{n} f(x)\mathrm{d}x,\quad n=1,2,\cdots.$

得 $\sum_{n=1}^{100} n^{-\frac{1}{2}} > \int_1^2 f(x)\mathrm{d}x+\int_2^3 f(x)\mathrm{d}x+\cdots+\int_{100}^{101} f(x)\mathrm{d}x$

$$=\int_1^{101} f(x)\mathrm{d}x > \int_1^{100} f(x)\mathrm{d}x = 2\sqrt{x}\,|_1^{100}=18,$$

$$\sum_{n=2}^{100} n^{-\frac{1}{2}} < \int_1^2 f(x)\mathrm{d}x+\int_2^3 f(x)\mathrm{d}x+\cdots+\int_{99}^{100} f(x)\mathrm{d}x=\int_1^{100} f(x)\mathrm{d}x=18,$$

$$\sum_{n=1}^{100} n^{-\frac{1}{2}} <1+\sum_{n=2}^{100} n^{-\frac{1}{2}} =19,\quad \text{即}18<\sum_{n=1}^{100} n^{-\frac{1}{2}}<19,\quad \text{所以}\sum_{n=1}^{100} n^{-\frac{1}{2}}\text{的整数部分为}18.$$

一、填空题

(本题共5个小题，每小题6分，共30分)

第5题



中国大学MOOC

(2017决赛,6分)曲线$L_1: y=\frac{1}{3}x^3+2x(0\le x\le 1)$绕直线$L_2: y=\frac{4}{3}x$旋转所生成的旋转曲面的面积为_____.

解　两平面P_s和$P_{s+\mathrm{d}s}$截曲面Σ的截下部分，这部分摊开可以看成一个细长条.这个细长条的长是$2\pi d$，宽是$\mathrm{d}s$，它的面积是$2\pi d\mathrm{d}s$.

即　　$\mathrm{d}S=2\pi d\mathrm{d}s$

曲线L_1上的点$(x_0,\frac{1}{3}x_0^3+2x_0)$到直线$L_2:\frac{4}{3}x-y=0$的距离为

$$d=\frac{\left|\frac{4}{3}x_0-(\frac{1}{3}{x_0}^3+2x_0)\right|}{\sqrt{1+(\frac{4}{3})^2}}=\frac{\left|{x_0}^3+2x_0\right|}{5},$$

曲线$L_1: y=\frac{1}{3}x^3+2x(0\le x\le 1)$绕直线$L_2: y=\frac{4}{3}x$旋转所生成的旋转曲面的面积为_____.

$$d=\frac{\left|x_0{}^3+2x_0\right|}{5},$$

由$y=\frac{1}{3}x^3+2x$,有$y'=x^2+2$，所以曲线$L_1: y=\frac{1}{3}x^3+2x$上的弧微分为

$$\mathrm{d}s=\sqrt{1+(y')^2}\,\mathrm{d}x=\sqrt{1+(x^2+2)^2}\,\mathrm{d}x$$

$$\mathrm{d}S=2\pi d\,\mathrm{d}s$$

$$S=\int_0^1 2\pi\frac{(x^3+2x)}{5}\sqrt{1+(x^2+2)^2}\,\mathrm{d}x=\frac{2\pi}{5}\int_0^1(x^3+2x)\sqrt{1+(x^2+2)^2}\,\mathrm{d}x$$

$$=\frac{2\pi}{5}\cdot\frac{1}{4}\int_0^1\sqrt{1+(x^2+2)^2}\,\mathrm{d}(x^2+2)^2$$

$$=\frac{\pi}{10}\cdot\frac{2}{3}\left[1+(x^2+2)^2\right]^{\frac{3}{2}}\Big|_0^1=\frac{\pi}{15}(10\sqrt{10}-5\sqrt{5})=\frac{\sqrt{5}(2\sqrt{2}-1)}{3}\pi.$$

二、（14分）



中国大学MOOC

(2017决赛,14分) 设$0<x<\frac{\pi}{2}$，证明: $\frac{4}{\pi^2}<\frac{1}{x^2}-\frac{1}{\tan^2 x}<\frac{2}{3}$.

分析 $\lim\limits_{x\to 0^+} f(x)=\frac{2}{3}$，$\lim\limits_{x\to \frac{\pi}{2}^-} f(x)=\frac{4}{\pi^2}$，证明$f(x)$在$0<x<\frac{\pi}{2}$单调递减

证 设$f(x)=\frac{1}{x^2}-\frac{1}{\tan^2 x}$ $(0<x<\frac{\pi}{2})$，$\left(\frac{1}{\tan^2 x}\right)'=\left(\frac{\cos^2 x}{\sin^2 x}\right)'=\left(\frac{1}{\sin^2 x}-1\right)'$

则 $f'(x)=-\frac{2}{x^3}+\frac{2\cos x}{\sin^3 x}=\frac{2(x^3\cos x-\sin^3 x)}{x^3\sin^3 x}$ ①

$x^3\cos x-\sin^3 x<0 \Leftrightarrow x^3-\frac{\sin^3 x}{\cos x}<0 \Leftrightarrow x<\frac{\sin x}{\sqrt[3]{\cos x}}$

$\Leftrightarrow \frac{\sin x}{\sqrt[3]{\cos x}}-x>0$ 令$\varphi(x)=\frac{\sin x}{\sqrt[3]{\cos x}}-x$ $(0<x<\frac{\pi}{2})$,

设$0<x<\frac{\pi}{2}$，证明：$\frac{4}{\pi^2}<\frac{1}{x^2}-\frac{1}{\tan^2 x}<\frac{2}{3}$.

令$\varphi(x)=\frac{\sin x}{\sqrt[3]{\cos x}}-x \quad (0<x<\frac{\pi}{2})$,

则$\varphi'(x)=\frac{\cos^{4/3}x+\frac{1}{3}\cos^{-2/3}x\cdot\sin^2 x}{\cos^{2/3}x}-1=\frac{\frac{2}{3}\cos^{4/3}x+\frac{1}{3}\cos^{-2/3}x}{\cos^{2/3}x}-1$

$=\frac{2}{3}\cos^{2/3}x+\frac{1}{3}\cos^{-4/3}x-1.$ 由均值不等式得

$\frac{2}{3}\cos^{2/3}x+\frac{1}{3}\cos^{-4/3}x=\frac{1}{3}(\cos^{2/3}x+\cos^{2/3}x+\cos^{-4/3}x)$

$>\sqrt[3]{\cos^{2/3}x\cdot\cos^{2/3}x\cdot\cos^{-4/3}x}=1$

$\frac{a+b+c}{3}>\sqrt[3]{abc}=1$

设$0<x<\frac{\pi}{2}$，证明：$\frac{4}{\pi^2}<\frac{1}{x^2}-\frac{1}{\tan^2 x}<\frac{2}{3}.$

$$f'(x)=\frac{2(x^3\cos x-\sin^3 x)}{x^3\sin^3 x} \qquad ①$$

$$\varphi'(x)=\frac{2}{3}\cos^{2/3}x+\frac{1}{3}\cos^{-4/3}x-1.$$

所以当$0<x<\frac{\pi}{2}$时，$\varphi'(x)>0$，从而$\varphi(x)$单调递增.

又$\varphi(0)=0$，因此$\varphi(x)>0$，即 $x^3\cos x-\sin^3 x<0.$

由①式得$f'(x)<0$，从而$f(x)$在区间$(0,\frac{\pi}{2})$单调递减.

设$0<x<\frac{\pi}{2}$，证明：$\frac{4}{\pi^2}<\frac{1}{x^2}-\frac{1}{\tan^2 x}<\frac{2}{3}$.

$f(x)$在区间$(0,\frac{\pi}{2})$单调递减

由于 $\lim\limits_{x\to\frac{\pi}{2}^-} f(x)=\lim\limits_{x\to\frac{\pi}{2}^-}(\frac{1}{x^2}-\frac{1}{\tan^2 x})=\lim\limits_{x\to\frac{\pi}{2}^-}(\frac{1}{x^2}-\frac{\cos^2 x}{\sin^2 x})=\frac{4}{\pi^2}$,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(\frac{1}{x^2}-\frac{1}{\tan^2 x})=\lim_{x\to 0^+}\frac{\tan^2 x-x^2}{x^4}$$

$$=\lim_{x\to 0^+}\frac{(\tan x+x)(\tan x-x)}{x^4}=\lim_{x\to 0^+}(\frac{\tan x+x}{x}\cdot\frac{\tan x-x}{x^3})=2\lim_{x\to 0^+}\frac{\tan x-x}{x^3}$$

$$=2\lim_{x\to 0^+}\frac{\sec^2 x-1}{3x^2}=\frac{2}{3}\lim_{x\to 0^+}\frac{\sin^2 x}{x^2\cos^2 x}=\frac{2}{3},$$

所以当$0<x<\frac{\pi}{2}$时，有 $\frac{4}{\pi^2}<\frac{1}{x^2}-\frac{1}{\tan^2 x}<\frac{2}{3}$.

设$0<x<\dfrac{\pi}{2}$，证明：$\dfrac{4}{\pi^2}<\dfrac{1}{x^2}-\dfrac{1}{\tan^2 x}<\dfrac{2}{3}$.

备注

令$\varphi(x)=\dfrac{\sin x}{\sqrt[3]{\cos x}}-x \quad (0<x<\dfrac{\pi}{2})$，则$\varphi'(x)=\dfrac{2}{3}\cos^{2/3}x+\dfrac{1}{3}\cos^{-4/3}x-1.$

由均值不等式得 $\dfrac{2}{3}\cos^{2/3}x+\dfrac{1}{3}\cos^{-4/3}x>\sqrt[3]{\cos^{2/3}x\cdot\cos^{2/3}x\cdot\cos^{-4/3}x}=1$

$\dfrac{a+b+c}{3}>\sqrt[3]{abc}=1$

所以当$0<x<\dfrac{\pi}{2}$时，$\varphi'(x)>0$，从而$\varphi(x)$单调递增.

令$t=\cos^{2/3}x$，设$g(t)=\dfrac{2}{3}t+\dfrac{1}{3}t^{-2}-1,$

则$g'(t)=\dfrac{2}{3}-\dfrac{2}{3}t^{-3}=\dfrac{2}{3}(1-t^{-3})<0,\quad t\in(0,1).$

有$g(t)>g(1)=0 \qquad \therefore \varphi'(x)>0$

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三、（14分）



中国大学MOOC

(2017决赛,14分)

设$f(x)$为$(-\infty,+\infty)$上周期为1的连续函数，且满足$0\le f(x)\le 1$与$\int_0^1 f(x)\mathrm{d}x=1$.

证明:$0\le x\le 13$时，有$\int_0^{\sqrt{x}} f(t)\mathrm{d}t+\int_0^{\sqrt{x+27}} f(t)\mathrm{d}t+\int_0^{\sqrt{13-x}} f(t)\mathrm{d}t\le 11$，并给出取等号的条件.

证 由条件$0\le f(x)\le 1$，有

$$\int_0^{\sqrt{x}} f(t)\mathrm{d}t+\int_0^{\sqrt{x+27}} f(t)\mathrm{d}t+\int_0^{\sqrt{13-x}} f(t)\mathrm{d}t\le \int_0^{\sqrt{x}}\mathrm{d}t+\int_0^{\sqrt{x+27}}\mathrm{d}t+\int_0^{\sqrt{13-x}}\mathrm{d}t$$

$$=\sqrt{x}+\sqrt{x+27}+\sqrt{13-x}=1\cdot\sqrt{x}+\sqrt{2}\cdot\sqrt{\frac{1}{2}(x+27)}+\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{2}(13-x)}$$

柯西不等式 $(\sum_{i=1}^{n}a_ib_i)^2\le\sum_{i=1}^{n}a_i^2\cdot\sum_{i=1}^{n}b_i^2$，当且仅当$a_i$与$b_i$对应成比例时等号成立.

证明：$\int_0^{\sqrt{x}} f(t)\mathrm{d}t+\int_0^{\sqrt{x+27}} f(t)\mathrm{d}t+\int_0^{\sqrt{13-x}} f(t)\mathrm{d}t \le 11$

$$\sqrt{x}+\sqrt{x+27}+\sqrt{13-x}=1\cdot\sqrt{x}+\sqrt{2}\cdot\sqrt{\frac{1}{2}(x+27)}+\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{2}(13-x)}$$

$$\le\sqrt{1+2+\frac{2}{3}}\cdot\sqrt{x+\frac{1}{2}(x+27)+\frac{3}{2}(13-x)}=11.$$

等号成立的充分必要条件是：$\sqrt{x}=\frac{1}{2}\sqrt{x+27}=\frac{3}{2}\sqrt{13-x}$，即$x=9$.

当$x=9$时，$\int_0^{\sqrt{x}} f(t)\mathrm{d}t+\int_0^{\sqrt{x+27}} f(t)\mathrm{d}t+\int_0^{\sqrt{13-x}} f(t)\mathrm{d}t$

$$=\int_0^{3} f(t)\mathrm{d}t+\int_0^{6} f(t)\mathrm{d}t+\int_0^{2} f(t)\mathrm{d}t=11\int_0^{1} f(t)\mathrm{d}t=11,$$

所以 $\int_0^{\sqrt{x}} f(t)\mathrm{d}t+\int_0^{\sqrt{x+27}} f(t)\mathrm{d}t+\int_0^{\sqrt{13-x}} f(t)\mathrm{d}t \le 11.$

四、（14分）



中国大学MOOC

(2017决赛, 14分) 设函数$f(x,y,z)$在区域$\Omega=\{(x,y,z)\,|\,x^2+y^2+z^2\le 1\}$上具有连续的二阶偏导数，且满足$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=\sqrt{x^2+y^2+z^2}$. 计算

$$I=\iiint_{\Omega}(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

解 记球面Σ：$x^2+y^2+z^2=1$外侧的单位法向量为$\vec{n}=\{\cos\alpha,\cos\beta,\cos\gamma\}$,

则 $$\frac{\partial f}{\partial \vec{n}}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma.$$

考虑曲面积分等式：$\iint_{\Sigma}\dfrac{\partial f}{\partial \vec{n}}\mathrm{d}S=\iint_{\Sigma}(x^2+y^2+z^2)\dfrac{\partial f}{\partial \vec{n}}\mathrm{d}S$ (1)

区域$\Omega=\{(x,y,z)\,|\,x^2+y^2+z^2\le 1\}$，$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=\sqrt{x^2+y^2+z^2}$.

计算$I=\iiint_{\Omega}(x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

考虑曲面积分等式：$\oiint_{\Sigma}\dfrac{\partial f}{\partial \vec{n}}\mathrm{d}S=\oiint_{\Sigma}(x^2+y^2+z^2)\dfrac{\partial f}{\partial \vec{n}}\mathrm{d}S$ (1)

$\dfrac{\partial f}{\partial \vec{n}}=\dfrac{\partial f}{\partial x}\cos\alpha+\dfrac{\partial f}{\partial y}\cos\beta+\dfrac{\partial f}{\partial z}\cos\gamma$.

对两边都利用高斯公式，得

左边$=\oiint_{\Sigma}\dfrac{\partial f}{\partial \vec{n}}\mathrm{d}S$

$=\oiint_{\Sigma}(\dfrac{\partial f}{\partial x}\cos\alpha+\dfrac{\partial f}{\partial y}\cos\beta+\dfrac{\partial f}{\partial z}\cos\gamma)\mathrm{d}S=\iiint_{\Omega}(\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2})\mathrm{d}V$ (2)

区域$\Omega=\{(x,y,z)\,|\,x^2+y^2+z^2\le 1\}$，$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=\sqrt{x^2+y^2+z^2}$.

计算$I=\iiint_{\Omega}(x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

$$\oiint_{\Sigma}\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=\oiint_{\Sigma}(x^2+y^2+z^2)\frac{\partial f}{\partial \vec{n}}\mathrm{d}S \qquad (1)$$

$$P(x,y,z)=(x^2+y^2+z^2)\frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{\partial P}{\partial x}=2x\frac{\partial f}{\partial x}+(x^2+y^2+z^2)\frac{\partial^2 f}{\partial x^2}$$

$$\text{左边}=\oiint_{\Sigma}\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=\iiint_{\Omega}(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2})\mathrm{d}V \qquad (2)$$

$$\text{右边}=\oiint_{\Sigma}(x^2+y^2+z^2)\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=\oiint_{\Sigma}(x^2+y^2+z^2)(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma)\mathrm{d}S$$

$$=2\iiint_{\Omega}(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z})\mathrm{d}V+\iiint_{\Omega}(x^2+y^2+z^2)(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2})\mathrm{d}V \qquad (3)$$

区域$\Omega=\{(x,y,z)\,|\,x^2+y^2+z^2\le 1\}$，$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=\sqrt{x^2+y^2+z^2}$.

计算$I=\iiint_\Omega (x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z})\mathrm{d}x\mathrm{d}y\mathrm{d}z$.　　$\oiint_\Sigma \dfrac{\partial f}{\partial \vec{n}}\mathrm{d}S=\oiint_\Sigma (x^2+y^2+z^2)\dfrac{\partial f}{\partial \vec{n}}\mathrm{d}S$　(1)

将(2)、(3)代入(1)得：$\iiint_\Omega (\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2})\mathrm{d}V$

$=2\iiint_\Omega (x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z})\mathrm{d}V+\iiint_\Omega (x^2+y^2+z^2)(\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2})\mathrm{d}V$

$\Rightarrow \iiint_\Omega \sqrt{x^2+y^2+z^2}\,\mathrm{d}V=2I+\iiint_\Omega (x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}\,\mathrm{d}V$

$I=\dfrac{1}{2}\iiint_\Omega [1-(x^2+y^2+z^2)]\sqrt{x^2+y^2+z^2}\,\mathrm{d}V=\dfrac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^1(1-r^2)r^3\mathrm{d}r=\dfrac{\pi}{6}$.

五、（14分）



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(2017决赛,14分)

设n阶方阵$\boldsymbol{A}$、$\boldsymbol{B}$满足$\boldsymbol{AB}=\boldsymbol{A}+\boldsymbol{B}$. 证明：若存在正整数$k$，使$\boldsymbol{A}^k=\boldsymbol{O}$ ($\boldsymbol{O}$为零矩阵)，则行列式$|\boldsymbol{B}+2017\boldsymbol{A}|=|\boldsymbol{B}|$.

分析　若$\boldsymbol{B}$可逆，$|\boldsymbol{B}+2017\boldsymbol{A}|=|\boldsymbol{B}(\boldsymbol{E}+2017\boldsymbol{B}^{-1}\boldsymbol{A})|=|\boldsymbol{B}|\cdot|\boldsymbol{E}+2017\boldsymbol{B}^{-1}\boldsymbol{A}|$.

证　由$\boldsymbol{AB}=\boldsymbol{A}+\boldsymbol{B}$得$(\boldsymbol{A}-\boldsymbol{E})(\boldsymbol{B}-\boldsymbol{E})=\boldsymbol{E}$,

则$(\boldsymbol{A}-\boldsymbol{E})(\boldsymbol{B}-\boldsymbol{E})=(\boldsymbol{B}-\boldsymbol{E})(\boldsymbol{A}-\boldsymbol{E})$,　化简可得到 $\boldsymbol{AB}=\boldsymbol{BA}$.

(1) 若$\boldsymbol{B}$可逆，则由$\boldsymbol{AB}=\boldsymbol{BA}$得$\boldsymbol{B}^{-1}\boldsymbol{A}=\boldsymbol{A}\boldsymbol{B}^{-1}$,

从而$(\boldsymbol{B}^{-1}\boldsymbol{A})^k=\boldsymbol{B}^{-1}\boldsymbol{A}\cdot\boldsymbol{B}^{-1}\boldsymbol{A}\cdots\cdots\boldsymbol{B}^{-1}\boldsymbol{A}=(\boldsymbol{B}^{-1})^k\boldsymbol{A}^k=\boldsymbol{O}$,

所以$\boldsymbol{B}^{-1}\boldsymbol{A}$的特征值全为0,　则$\boldsymbol{E}+2017\boldsymbol{B}^{-1}\boldsymbol{A}$的特征值为1,

设n阶方阵$\boldsymbol{A}$、$\boldsymbol{B}$满足$\boldsymbol{AB}=\boldsymbol{A}+\boldsymbol{B}$. 证明：若存在正整数$k$，使$\boldsymbol{A}^k=\boldsymbol{O}$（$\boldsymbol{O}$为零矩阵），则行列式$|\boldsymbol{B}+2017\boldsymbol{A}|=|\boldsymbol{B}|$.

所以$\boldsymbol{B}^{-1}\boldsymbol{A}$的特征值全为$\mathbf{0}$，　则$\boldsymbol{E}+\mathbf{2017}\boldsymbol{B}^{-1}\boldsymbol{A}$的特征值为$\mathbf{1}$，

因此$\left|\boldsymbol{E}+\mathbf{2017}\boldsymbol{B}^{-1}\boldsymbol{A}\right|=\mathbf{1}$，　$|\boldsymbol{E}+2017\boldsymbol{A}|=|\boldsymbol{B}|\cdot\left|\boldsymbol{E}+\mathbf{2017}\boldsymbol{B}^{-1}\boldsymbol{A}\right|=|\boldsymbol{B}|$.

(2) 若$\boldsymbol{B}$不可逆，则存在无穷多个数$\boldsymbol{t}$，使$\boldsymbol{B}_t=\boldsymbol{tE}+\boldsymbol{B}$可逆，且有$\boldsymbol{AB}_t=\boldsymbol{B}_t\boldsymbol{A}$.

利用**(1)**的结论，有恒等式$|\boldsymbol{B}_t+\mathbf{2017}\boldsymbol{A}|=|\boldsymbol{B}_t|$.

取$\boldsymbol{t}=\mathbf{0}$，得$|\boldsymbol{B}+\mathbf{2017}\boldsymbol{A}|=|\boldsymbol{B}|$.

六、（14分）



中国大学MOOC

(2017决赛,14分) 设$a_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n$. (1) 证明：极限$\lim_{n\to\infty}a_n$存在；

(2) 记$\lim_{n\to\infty}a_n=C$，讨论级数$\sum_{n=1}^{\infty}(a_n-C)$的敛散性.

解 (1)利用不等式：当$x>0$时,$\frac{x}{1+x}<\ln(1+x)<x$.

$$a_n-a_{n-1}=\frac{1}{n}-\ln\frac{n}{n-1}=\frac{1}{n}-\ln\left(1+\frac{1}{n-1}\right)\le\frac{1}{n}-\frac{\frac{1}{n-1}}{1+\frac{1}{n-1}}=0,$$

$$a_n=\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=2}^{n}\ln\frac{k}{k-1}=1+\sum_{k=2}^{n}\left(\frac{1}{k}-\ln\frac{k}{k-1}\right)$$

$$=1+\sum_{k=2}^{n}\left[\frac{1}{k}-\ln\left(1+\frac{1}{k-1}\right)\right]\ge1+\sum_{k=2}^{n}\left(\frac{1}{k}-\frac{1}{k-1}\right)=1+\left(\frac{1}{n}-1\right)=\frac{1}{n}>0,$$

设$a_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n$. (1) 证明：极限$\lim_{n\to\infty}a_n$存在；

(2) 记$\lim_{n\to\infty}a_n=C$，讨论级数$\sum_{n=1}^{\infty}(a_n-C)$的敛散性.

$$a_n=1+\sum_{k=2}^{n}\left(\frac{1}{k}-\ln\frac{k}{k-1}\right)$$

$a_n-a_{n-1}\le 0$，$a_n>0$

所以$\{a_n\}$单调减少有下界，故$\lim_{n\to\infty}a_n$存在.

(2) 显然以a_n为部分和的级数为$1+\sum_{n=2}^{\infty}\left[\frac{1}{n}-\ln n+\ln(n-1)\right]$,

则该级数收敛于C，且$a_n-C>0$. 用r_n记该级数的余项，则

$$a_n-C=-r_n=-\sum_{k=n+1}^{\infty}\left[\frac{1}{k}-\ln k+\ln(k-1)\right]=\sum_{k=n+1}^{\infty}\left[\ln\left(1+\frac{1}{k-1}\right)-\frac{1}{k}\right].$$

根据泰勒公式，当$x>0$时,$\ln(1+x)>x-\frac{x^2}{2}$,

设$a_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n$. (2) 记$\lim_{n\to\infty}a_n=C$，讨论级数$\sum_{n=1}^{\infty}(a_n-C)$的敛散性.

$$a_n-C=\sum_{k=n+1}^{\infty}\left[\ln\left(1+\frac{1}{k-1}\right)-\frac{1}{k}\right]$$

所以$a_n-C>\sum_{k=n+1}^{\infty}\left[\frac{1}{k-1}-\frac{1}{2(k-1)^2}-\frac{1}{k}\right]$.

记$b_n=\sum_{k=n+1}^{\infty}\left[\frac{1}{k-1}-\frac{1}{2(k-1)^2}-\frac{1}{k}\right]$，下面证明正项级数$\sum_{n=1}^{\infty}b_n$发散.

$$c_n\overset{\Delta}{=}n\sum_{k=n+1}^{\infty}\left[\frac{1}{k-1}-\frac{1}{k}-\frac{1}{2(k-1)(k-2)}\right]<nb_n<n\sum_{k=n+1}^{\infty}\left[\frac{1}{k-1}-\frac{1}{k}-\frac{1}{2k(k-1)}\right]\overset{\Delta}{=}d_n,$$

而当$n\to\infty$时，$c_n=\frac{n-2}{2(n-1)}\to\frac{1}{2}$，同理当$n\to\infty$时，$d_n\to\frac{1}{2}$，所以$\lim_{n\to\infty}nb_n=\frac{1}{2}$.

由$\sum_{n=1}^{\infty}\frac{1}{n}$发散, 根据比较判别法可知，级数$\sum_{n=1}^{\infty}b_n$发散. 因此级数$\sum_{n=1}^{\infty}(a_n-C)$发散.

END



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