全国大学生
数学竞赛



第11届决赛

中国大学MOOC

一、填空题

本题满分30分，每小题6分

第1题



中国大学MOOC

2021年4月决赛，6分 $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{(1-\sqrt{\sin x})(1-\sqrt[3]{\sin x})\cdots(1-\sqrt[n]{\sin x})}{(1-\sin x)^{n-1}}=$ _____. $\dfrac{0}{0}$

解

$$I=\lim_{x\to\frac{\pi}{2}}\frac{1-\sqrt{\sin x}}{1-\sin x}\cdot\frac{1-\sqrt[3]{\sin x}}{1-\sin x}\cdots\cdots\frac{1-\sqrt[n]{\sin x}}{1-\sin x}$$

$t\to 0$时，$\sqrt[n]{1+t}-1\sim\dfrac{1}{n}t$

$$=\lim_{x\to\frac{\pi}{2}}\frac{1-\sqrt{1+(\sin x-1)}}{1-\sin x}\cdot\lim_{x\to\frac{\pi}{2}}\frac{1-\sqrt[3]{1+(\sin x-1)}}{1-\sin x}\cdots\cdots\lim_{x\to\frac{\pi}{2}}\frac{1-\sqrt[n]{1+(\sin x-1)}}{1-\sin x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{1+(\sin x-1)}-1}{\sin x-1}\cdot\lim_{x\to\frac{\pi}{2}}\frac{\sqrt[3]{1+(\sin x-1)}-1}{\sin x-1}\cdots\cdots\lim_{x\to\frac{\pi}{2}}\frac{\sqrt[n]{1+(\sin x-1)}-1}{\sin x-1}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{2}(\sin x-1)}{\sin x-1}\cdot\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{3}(\sin x-1)}{\sin x-1}\cdots\cdots\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{n}(\sin x-1)}{\sin x-1}=\frac{1}{2}\cdot\frac{1}{3}\cdots\cdots\frac{1}{n}=\frac{1}{n!}.$$

一、填空题

本题满分30分，每小题6分

第2题



中国大学MOOC

2021年4月决赛，6分

设函数$y=f(x)$由方程$3x-y=2\arctan(y-2x)$所确定，则曲线$y=f(x)$在点$P(1+\frac{\pi}{2},3+\pi)$处的切线方程为_____.

$y-y_0=f'(x_0)(x-x_0)$

解 对方程$3x-y=2\arctan(y-2x)$两边求导得 $3-y'=2\dfrac{y'-2}{1+(y-2x)^2}$

将点$P(1+\frac{\pi}{2},3+\pi)$带入上式得 $y'|_P=\dfrac{5}{2}$,

故曲线$y=f(x)$在点P处的切线方程为 $y-(3+\pi)=\dfrac{5}{2}(x-1-\dfrac{\pi}{2})$,

即 $y=\dfrac{5}{2}x+\dfrac{1}{2}-\dfrac{\pi}{4}$.

一、填空题

本题满分30分，每小题6分

第3题



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2021年4月决赛，6分

设平面曲线L的方程为$Ax^2+By^2+Cxy+Dx+Ey+F=0$，且通过五个点$P_1(-1,0)$、$P_2(0,-1)$、$P_3(0,1)$、$P_4(2,-1)$和$P_5(2,1)$，则L上任意两点之间的直线距离最大值为_____.

解 将所给点代入方程得
$$\begin{cases} A-D+F=0 \\ B-E+F=0 \\ B+E+F=0 \\ 4A+B-2C+2D-E+F=0 \\ 4A+B+2C+2D+E+F=0 \end{cases} \Rightarrow \begin{cases} A=1\,(\text{令}) \\ B=3 \\ C=0 \\ D=-2 \\ E=0 \\ F=-3 \end{cases}$$

解得曲线L的方程为$x^2+3y^2-2x-3=0$，其标准型为$\dfrac{(x-1)^2}{4}+\dfrac{y^2}{\frac{4}{3}}=1$,

因此曲线L上两点间的最长直线距离为4.

一、填空题

本题满分30分，每小题6分

第4题



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2021年4月决赛，6分

设$f(x)=(x^2+2x-3)^n\arctan^2\frac{x}{3}$，其中$n$为正整数，则$f^{(n)}(-3)=$_____.

分析 $f(x)=(x+3)^n(x-1)^n\arctan^2\frac{x}{3}=(x+3)^n g(x).$

解 记$g(x)=(x-1)^n\arctan^2\frac{x}{3}$，则$f(x)=(x+3)^n g(x)$,

利用莱布尼兹公式得 $f^{(n)}(x)=n!g(x)+\sum_{k=0}^{n-1}C_n^k[(x+3)^n]^{(k)}g^{(n-k)}(x)$

则 $f^{(n)}(-3)=n!g(-3)=(-1)^n 4^{n-2}n!\pi^2.$

莱布尼兹公式 $(uv)^{(n)}=u^{(n)}v+nu^{(n-1)}v'+\frac{1}{2!}n(n-1)u^{(n-2)}v''+\cdots$

$+\frac{n(n-1)\cdots(n-k+1)}{k!}u^{(n-k)}v^{(k)}+\cdots+uv^{(n)}$

一、填空题

本题满分30分，每小题6分

第5题



中国大学MOOC

2021年4月决赛，6分

设函数$f(x)$的导数$f'(x)$在$[0,1]$上连续，$f(0)=f(1)=0$，且满足

$$\int_0^1[f'(x)]^2\mathrm{d}x-8\int_0^1 f(x)\mathrm{d}x+\frac{4}{3}=0,\ \text{则}f(x)=_____.$$

解 因为 $\int_0^1 f(x)\mathrm{d}x=xf(x)\big|_0^1-\int_0^1 xf'(x)\mathrm{d}x=-\int_0^1 xf'(x)\mathrm{d}x$,

$f'(x)+\square\square=0$

$$\int_0^1 f'(x)\mathrm{d}x=f(x)\big|_0^1=0,\quad \int_0^1(4x^2-4x+1)\mathrm{d}x=\frac{1}{3},$$

$$\int_0^1[f'(x)]^2\mathrm{d}x-8\int_0^1 f(x)\mathrm{d}x+\frac{4}{3}=\int_0^1\left[f'^2(x)-8f(x)+\frac{4}{3}\right]\mathrm{d}x=\int_0^1[f'(x)+\square\square]^2\mathrm{d}x=0$$

$$=\int_0^1[f'^2(x)+8xf'(x)-4f'(x)+(16x^2-16x+4)]\mathrm{d}x=\int_0^1[f'(x)+4x-2]^2\mathrm{d}x=0$$

因此 $f'(x)=2-4x,\ \Rightarrow f(x)=2x-2x^2+C,$

由 $f(0)=0$ 得 $C=0,$ 故 $f(x)=2x-2x^2.$

二、（12分）



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2021年4月决赛，12分　求极限 $\lim\limits_{n\to\infty}\sqrt{n}\left(1-\sum\limits_{k=1}^{n}\dfrac{1}{n+\sqrt{k}}\right)$.

解

$$a_n=\sqrt{n}\sum_{k=1}^{n}\left(\frac{1}{n}-\frac{1}{n+\sqrt{k}}\right)=\sqrt{n}\sum_{k=1}^{n}\frac{\sqrt{k}}{n(n+\sqrt{k})}=\sum_{k=1}^{n}\frac{\sqrt{k}}{\sqrt{n}(n+\sqrt{k})}$$

$$\le\sum_{k=1}^{n}\frac{\sqrt{k}}{n\sqrt{n}}=\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\sqrt{k}$$

$$\sqrt{k}=\int_k^{k+1}\sqrt{k}\,\mathrm{d}x\le\int_k^{k+1}\sqrt{x}\,\mathrm{d}x$$

因为 $\sum\limits_{k=1}^{n}\sqrt{k}\le\sum\limits_{k=1}^{n}\int_k^{k+1}\sqrt{x}\,\mathrm{d}x=\int_1^{n+1}\sqrt{x}\,\mathrm{d}x=\dfrac{2}{3}x^{\frac{3}{2}}\Big|_1^{n+1}=\dfrac{2}{3}\left[(n+1)\sqrt{n+1}-1\right]$

所以 $a_n\le\dfrac{1}{n\sqrt{n}}\sum\limits_{k=1}^{n}\sqrt{k}\le\dfrac{1}{n\sqrt{n}}\cdot\dfrac{2}{3}\left[(n+1)\sqrt{n+1}-1\right]$

$$<\frac{2}{3}\cdot\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}}=\frac{2}{3}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{1}{n}}.$$

求极限 $\lim\limits_{n\to\infty}\sqrt{n}\left(1-\sum\limits_{k=1}^{n}\dfrac{1}{n+\sqrt{k}}\right)$.

接上

$$\text{又}\sum_{k=1}^{n}\sqrt{k}\geq\sum_{k=1}^{n}\int_{k-1}^{k}\sqrt{x}\mathrm{d}x=\int_{0}^{n}\sqrt{x}\mathrm{d}x=\frac{2}{3}n\sqrt{n},$$

$$\text{得}\ a_n\geq\frac{1}{\sqrt{n}(n+\sqrt{n})}\sum_{k=1}^{n}\sqrt{k}\geq\frac{2}{3}\cdot\frac{n}{n+\sqrt{n}}.$$

$$\sum_{k=1}^{n}\sqrt{k}\leq\sum_{k=1}^{n}\int_{k}^{k+1}\sqrt{x}\mathrm{d}x$$

$$a_n=\sum_{k=1}^{n}\frac{\sqrt{k}}{\sqrt{n}(n+\sqrt{k})}$$

$$\text{于是}\quad\frac{2}{3}\cdot\frac{n}{n+\sqrt{n}}\leq a_n<\frac{2}{3}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{2}{3}\cdot\frac{n}{n+\sqrt{n}}=\frac{2}{3},\qquad\lim_{n\to\infty}\frac{2}{3}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{1}{n}}=\frac{2}{3}$$

由夹逼准则得 $\lim\limits_{n\to\infty}\sqrt{n}\left(1-\sum\limits_{k=1}^{n}\dfrac{1}{n+\sqrt{k}}\right)=\lim\limits_{n\to\infty}a_n=\dfrac{2}{3}$.

三、（12分）



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2021年4月决赛，12分

设$F(x_1,x_2,x_3)=\int_0^{2\pi} f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\mathrm{d}\varphi$，其中$f(u,v)$具有二阶连续偏导数. 已知

$$\frac{\partial F}{\partial x_i}=\int_0^{2\pi}\frac{\partial}{\partial x_i}\left[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\right]\mathrm{d}\varphi,\quad \frac{\partial^2 F}{\partial x_i^2}=\int_0^{2\pi}\frac{\partial^2}{\partial x_i^2}\left[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\right]\mathrm{d}\varphi,$$

$i=1,2,3$，试求$x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}$并要求化简.

解 令$u=x_1+x_3\cos\varphi, v=x_2+x_3\sin\varphi$，则$f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)=f(u,v)$

根据复合函数求导法则得$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial u}\cdot 1+\frac{\partial f}{\partial v}\cdot 0=\frac{\partial f}{\partial u}$,

$$\frac{\partial f}{\partial x_2}=\frac{\partial f}{\partial u}\cdot 0+\frac{\partial f}{\partial v}\cdot 1=\frac{\partial f}{\partial v},\qquad \frac{\partial f}{\partial x_3}=\frac{\partial f}{\partial u}\cdot\cos\varphi+\frac{\partial f}{\partial v}\cdot\sin\varphi,$$

试求 $x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}$ 并要求化简.

接上

$$f(x_1+x_3\cos\varphi, x_2+x_3\sin\varphi)=f(u,v)$$

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial u},\quad \frac{\partial f}{\partial x_2}=\frac{\partial f}{\partial v},\quad \frac{\partial f}{\partial x_3}=\frac{\partial f}{\partial u}\cdot\cos\varphi+\frac{\partial f}{\partial v}\cdot\sin\varphi,$$

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\cdot 1+\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\cdot 0=\frac{\partial^2 f}{\partial u^2},$$

$$\frac{\partial^2 f}{\partial x_2^2}=\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial v}\right)=\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\cdot 0+\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\cdot 1=\frac{\partial^2 f}{\partial v^2},$$

试求 $x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}$ 并要求化简.

接上

$$f(x_1+x_3\cos\varphi, x_2+x_3\sin\varphi)=f(u,v)$$

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial u},\quad \frac{\partial f}{\partial x_2}=\frac{\partial f}{\partial v},\quad \frac{\partial f}{\partial x_3}=\frac{\partial f}{\partial u}\cdot\cos\varphi+\frac{\partial f}{\partial v}\cdot\sin\varphi,$$

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial^2 f}{\partial u^2},\quad \frac{\partial^2 f}{\partial x_2^2}=\frac{\partial^2 f}{\partial v^2},$$

$$\frac{\partial^2 f}{\partial x_3^2}=\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u}\cdot\cos\varphi+\frac{\partial f}{\partial v}\cdot\sin\varphi\right)=\cos\varphi\cdot\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u}\right)+\sin\varphi\cdot\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial v}\right)$$

$$=\cos\varphi\cdot\left(\frac{\partial^2 f}{\partial u^2}\cdot\cos\varphi+\frac{\partial^2 f}{\partial u\partial v}\cdot\sin\varphi\right)+\sin\varphi\cdot\left(\frac{\partial^2 f}{\partial v\partial u}\cdot\cos\varphi+\frac{\partial^2 f}{\partial v^2}\cdot\sin\varphi\right)$$

$$=\frac{\partial^2 f}{\partial u^2}\cdot\cos^2\varphi+\frac{\partial^2 f}{\partial u\partial v}\cdot\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cdot\sin^2\varphi,$$

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$$\frac{\partial^2 F}{\partial x_i^2}=\int_0^{2\pi}\frac{\partial^2}{\partial x_i^2}\left[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\right]\mathrm{d}\varphi,\quad i=1,2,3,\quad \text{试求 } x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}$$

接上

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial^2 f}{\partial u^2}\qquad \frac{\partial^2 f}{\partial x_2^2}=\frac{\partial^2 f}{\partial v^2}\qquad \frac{\partial^2 f}{\partial x_3^2}=\frac{\partial^2 f}{\partial u^2}\cdot\cos^2\varphi+\frac{\partial^2 f}{\partial u\partial v}\cdot\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cdot\sin^2\varphi$$

$$x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)$$

$$=x_3\left[\int_0^{2\pi}\frac{\partial^2 f}{\partial u^2}\mathrm{d}\varphi+\int_0^{2\pi}\frac{\partial^2 f}{\partial v^2}\mathrm{d}\varphi-\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\cos^2\varphi+\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\sin^2\varphi\right)\mathrm{d}\varphi\right],$$

$$=x_3\left[\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cos^2\varphi\right)\mathrm{d}\varphi\right]$$

$$\frac{\partial F}{\partial x_i}=\int_0^{2\pi}\frac{\partial}{\partial x_i}\left[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\right]\mathrm{d}\varphi,\quad \text{求}\ x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}.$$

接上

$$x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)=x_3\left[\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cos^2\varphi\right)\mathrm{d}\varphi\right]$$

$$\frac{\partial f}{\partial x_3}=\frac{\partial f}{\partial u}\cdot\cos\varphi+\frac{\partial f}{\partial v}\cdot\sin\varphi$$

$$f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)=f(u,v)$$

$$\frac{\partial F}{\partial x_3}=\int_0^{2\pi}\left(\cos\varphi\frac{\partial f}{\partial u}+\sin\varphi\frac{\partial f}{\partial v}\right)\mathrm{d}\varphi=\int_0^{2\pi}\frac{\partial f}{\partial u}\mathrm{d}\sin\varphi-\int_0^{2\pi}\frac{\partial f}{\partial v}\mathrm{d}\cos\varphi$$

$$=\frac{\partial f}{\partial u}\cdot\sin\varphi\Big|_0^{2\pi}-\int_0^{2\pi}\sin\varphi\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial f}{\partial u}\right)\right]\mathrm{d}\varphi-\left[\frac{\partial f}{\partial v}\cdot\cos\varphi\Big|_0^{2\pi}-\int_0^{2\pi}\cos\varphi\cdot\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial f}{\partial v}\right)\right]\mathrm{d}\varphi\right]$$

$$=-\int_0^{2\pi}\sin\varphi\left(\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial\varphi}+\frac{\partial^2 f}{\partial u\partial v}\cdot\frac{\partial v}{\partial\varphi}\right)\mathrm{d}\varphi+\int_0^{2\pi}\cos\varphi\left(\frac{\partial^2 f}{\partial v\partial u}\cdot\frac{\partial u}{\partial\varphi}+\frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial v}{\partial\varphi}\right)\mathrm{d}\varphi$$

求 $x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}$

$$\frac{\partial u}{\partial \varphi}=-x_3\sin\varphi,\quad \frac{\partial v}{\partial \varphi}=x_3\cos\varphi$$

接上

$$x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)=x_3\left[\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cos^2\varphi\right)\mathrm{d}\varphi\right]$$

$$\frac{\partial F}{\partial x_3}=-\int_0^{2\pi}\sin\varphi\left(\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial \varphi}+\frac{\partial^2 f}{\partial u\partial v}\cdot\frac{\partial v}{\partial \varphi}\right)\mathrm{d}\varphi+\int_0^{2\pi}\cos\varphi\left(\frac{\partial^2 f}{\partial v\partial u}\cdot\frac{\partial u}{\partial \varphi}+\frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial v}{\partial \varphi}\right)\mathrm{d}\varphi$$

$$=x_3\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{1}{2}\sin 2\varphi\frac{\partial^2 f}{\partial u\partial v}\right)\mathrm{d}\varphi+x_3\int_0^{2\pi}\left(-\frac{1}{2}\sin 2\varphi\frac{\partial^2 f}{\partial u\partial v}+\frac{\partial^2 f}{\partial v^2}\cos^2\varphi\right)\mathrm{d}\varphi$$

$$=x_3\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cos^2\varphi\right)\mathrm{d}\varphi$$

所以 $x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}=0.$

四、（10分）



中国大学MOOC

例 2021年4月决赛，10分

设函数$f(x)$在$[0,1]$上具有连续导数，且$\int_0^1 f(x)\mathrm{d}x=\frac{5}{2}$，$\int_0^1 xf(x)\mathrm{d}x=\frac{3}{2}$.

证明：存在$\xi\in(0,1)$，使得$f'(\xi)=3$.

分析 $\int_0^1 x(1-x)[3-f'(x)]\mathrm{d}x=0 \Rightarrow \xi(1-\xi)[3-f'(\xi)]=0 \Rightarrow f'(\xi)=3.$

证 利用分部积分公式，

$$\int_0^1 x(1-x)[3-f'(x)]\mathrm{d}x=\int_0^1 x(1-x)\mathrm{d}[3x-f(x)]$$

$$=x(1-x)[3x-f(x)]\Big|_0^1-\int_0^1(1-2x)[3x-f(x)]\mathrm{d}x=\int_0^1 3x(2x-1)\mathrm{d}x+\int_0^1(1-2x)f(x)\mathrm{d}x$$

$$=\left(2x^3-\frac{3}{2}x^2\right)\Big|_0^1+\int_0^1 f(x)\mathrm{d}x-2\int_0^1 xf(x)\mathrm{d}x=2-\frac{3}{2}+\frac{5}{2}-3=0$$

根据积分中值定理存在$\xi\in(0,1)$，使得$\xi(1-\xi)[3-f'(\xi)]=0$，即$f'(\xi)=3$.

五、（12分）



中国大学MOOC

2021年4月决赛，12分

设B_1，B_2,…，B_{2021}为空间R^3中半径不为零的2021个球，$A=(a_{ij})$为2021阶方阵，其(i,j)元a_{ij}为球B_i与B_j相交部分的体积.证明：行列式$|E+A|>1$，其中E为单位矩阵.

证 记Ω为以原点O为球心且包含B_1，B_2,…，B_{2021}在内的球，考察二次型 $f=\sum_{i=1}^{2021}\sum_{j=1}^{2021}a_{ij}z_iz_j$，

$$a_{ij}=\iiint_{\Omega}\varphi_i(t,u,v)\varphi_j(t,u,v)\mathrm{d}t\mathrm{d}u\mathrm{d}v,\quad \text{其中}\varphi_i(t,u,v)=\begin{cases}1,(t,u,v)\in B_i\\0,(t,u,v)\in\Omega\setminus B_i\end{cases}$$

$$f=\sum_{i=1}^{2021}\sum_{j=1}^{2021}a_{ij}z_iz_j=\sum_{i=1}^{2021}\sum_{j=1}^{2021}\iiint_{\Omega}\left[\varphi_i(t,u,v)z_i\right]\left[\varphi_j(t,u,v)z_j\right]\mathrm{d}t\mathrm{d}u\mathrm{d}v$$

$$=\iiint_{\Omega}\left[\sum_{i=1}^{2021}\varphi_i(t,u,v)z_i\right]\cdot\left[\sum_{j=1}^{2021}\varphi_j(t,u,v)z_j\right]\mathrm{d}t\mathrm{d}u\mathrm{d}v$$

$$=\iiint_{\Omega}\left[\sum_{i=1}^{2021}\varphi_i(t,u,v)z_i\right]\cdot\left[\sum_{i=1}^{2021}\varphi_i(t,u,v)z_i\right]\mathrm{d}t\mathrm{d}u\mathrm{d}v=\iiint_{\Omega}\left[\sum_{i=1}^{2021}\varphi_i(t,u,v)z_i\right]^2\mathrm{d}t\mathrm{d}u\mathrm{d}v\geq 0$$

设B_1，B_2,…，B_{2021}为空间R^3中半径不为零的2021个球，$A=(a_{ij})$为2021阶方阵，其(i,j)元a_{ij}为球B_i与B_j相交部分的体积.证明：行列式$|E+A|>1$，其中E为单位矩阵.

接上

$$f=\iiint_{\Omega}\sum_{i=1}^{2021}\left[\varphi_i(t,u,v)z_i\right]^2\mathrm{d}t\mathrm{d}u\mathrm{d}v\geq 0$$

另一方面，存在正交变换$Z=PY$使得$f=\lambda_1y_1^2+\lambda_2y_2^2+\cdots\lambda_{2021}y_{2021}^2$，其中$\lambda_1$，$\lambda_2$,…，$\lambda_{2021}$为$A$的全部特征值，

因为二次型$f\geq 0$，所以A的特征值$\lambda_i\geq 0\ (i=1,2,\cdots,2021)$.

于是$|E+A|=|P^{-1}(E+\Lambda)P|=|P^{-1}|\cdot|E+\Lambda|\cdot|P|=|E+\Lambda|$

$$=(1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_{2021})\geq 1,$$

注意到A不是零矩阵，所以至少有一个特征值$\lambda_i>0$，　故$|E+A|>1$.

六、（12分）



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2021年4月决赛，12分

设Ω是由光滑的简单封闭曲面Σ围成的有界闭区域，函数$f(x,y,z)$在Ω上具有连续二阶偏导数，且$f(x,y,z)|_{(x,y,z)\in\Sigma}=0$. 记$\nabla f$为$f(x,y,z)$的梯度，并令$\Delta f=\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}$.

证明：对任意常数$C>0$，恒有$C\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+\dfrac{1}{C}\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\geq 2\iiint_\Omega|\nabla f|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

证 首先利用Guass公式，Σ取外侧.

$$\oiint_\Sigma f\frac{\partial f}{\partial x}\mathrm{d}y\mathrm{d}z+f\frac{\partial f}{\partial y}\mathrm{d}z\mathrm{d}x+f\frac{\partial f}{\partial z}\mathrm{d}x\mathrm{d}y$$

$$=\iiint_\Omega\left(f\Delta f+|\nabla f|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$f(x,y,z)|_{(x,y,z)\in\Sigma}=0$ 等于零

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(f\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial f}{\partial z}\right)$$

$$=\left(\frac{\partial f}{\partial x}\right)^2+f\frac{\partial^2 f}{\partial x^2}+\left(\frac{\partial f}{\partial y}\right)^2+f\frac{\partial^2 f}{\partial y^2}+\left(\frac{\partial f}{\partial z}\right)^2+f\frac{\partial^2 f}{\partial z^2}=f\Delta f+|\nabla f|^2$$

等于零

$$\nabla f=\left\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\}$$

证明：$C\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+\frac{1}{C}\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\geq 2\iiint_\Omega|\nabla f|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$

接上

$$\iint_\Sigma f\frac{\partial f}{\partial x}\mathrm{d}y\mathrm{d}z+f\frac{\partial f}{\partial y}\mathrm{d}z\mathrm{d}x+f\frac{\partial f}{\partial z}\mathrm{d}x\mathrm{d}y=2\iiint_\Omega\left(f\Delta f+|\nabla f|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=0$$

$$\iiint_\Omega|\nabla f|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint_\Omega(-f\Delta f)\mathrm{d}x\mathrm{d}y\mathrm{d}z\leq\sqrt{\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z}\cdot\sqrt{\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z}$$

Cauchy不等式

对任意常数$C>0$，恒有

$$C\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+\frac{1}{C}\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\geq 2\sqrt{\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z}\cdot\sqrt{\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z}$$

均值不等式

$\geq 2\iiint_\Omega|\nabla f|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$　结论得证.

七、（12分）



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设$\{u_n\}$是正数列，满足$\frac{u_{n+1}}{u_n}=1-\frac{\alpha}{n}+O(\frac{1}{n^\beta})$，其中常数$\alpha>0$，$\beta>1$.

(1) 对于$v_n=n^\alpha u_n$，判断级数$\sum_{n=1}^{\infty}\ln\frac{v_{n+1}}{v_n}$的敛散性；(2) 讨论级数$\sum_{n=1}^{\infty}u_n$的敛散性.

[注：设数列$\{a_n\},\{b_n\}$满足$\lim_{n\to\infty}a_n=0,\lim_{n\to\infty}b_n=0$，则$a_n=O(b_n)$

$\Leftrightarrow$ 存在$M>0$及正整数N，使得$|a_n|\le M|b_n|$对任意$n>N$成立.]

解 (1) $\ln\frac{v_{n+1}}{v_n}=\ln\frac{(n+1)^\alpha u_{n+1}}{n^\alpha u_n}=\alpha\ln\left(1+\frac{1}{n}\right)+\ln\left[1-\frac{\alpha}{n}+O(\frac{1}{n^\beta})\right]$ $\ln(1+u)=u+O(u^2)$

$=\left[\frac{\alpha}{n}+O(\frac{1}{n^2})\right]+\left[-\frac{\alpha}{n}+O(\frac{1}{n^2})+O(\frac{1}{n^\beta})\right]=O(\frac{1}{n^2})+O(\frac{1}{n^\beta})=O(\frac{1}{n^\gamma})$，其中$\gamma=\min\{2,\beta\}>1$，

故$\exists C>0$及正整数N，使$\left|\ln\frac{v_{n+1}}{v_n}\right|\le C\left|\frac{1}{n^\gamma}\right|$对$\forall n>N$成立，所以级数$\sum_{n=1}^{\infty}\ln\frac{v_{n+1}}{v_n}$绝对收敛；

设$\{u_n\}$是正数列，满足$\frac{u_{n+1}}{u_n}=1-\frac{\alpha}{n}+O(\frac{1}{n^\beta})$

(1) 对于$v_n=n^\alpha u_n$，判断级数$\sum_{n=1}^{\infty}\ln\frac{v_{n+1}}{v_n}$的敛散性；(2)讨论级数$\sum_{n=1}^{\infty}u_n$的敛散性.

接上

(1) 级数$\sum_{n=1}^{\infty}\ln\frac{v_{n+1}}{v_n}$收敛；

$\frac{u_n}{\frac{1}{n^\alpha}}=v_n$

(2) 因为$\sum_{k=1}^{n}\ln\frac{v_{k+1}}{v_k}=\ln v_{n+1}-\ln v_1$，所以由(1)的结论可知，极限$\lim_{n\to\infty}\ln v_n$存在，

令$\lim_{n\to\infty}\ln v_n=a$，则$\lim_{n\to\infty}v_n=e^a>0$，$\lim_{n\to\infty}\frac{u_n}{\frac{1}{n^\alpha}}=\lim_{n\to\infty}v_n=e^a>0.$

根据正项级数的比较判别法，级数$\sum_{n=1}^{\infty}u_n$当$\alpha>1$时收敛，$\alpha\le 1$时发散.

END



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