SEMINAR: ABUNDANCE IN CHAR p

Background: In char p, MMP for threefold when $p \geq 5$ has been established by Hacon-Xu, Birkar (-Waldron) and Hacon-Witaszek (the case $p = 5$), abundance was proved by L. Zhang for minimal X with $q(X) > 0$ and nonvanishing was proved by Xu-Zhang (the argument applies when $p \geq 5$).

This seminar aims to discuss abundance in char p. The strategy are almost the same with that adopted in char 0, but one does not need to read the proof in char 0, the argument in char p is self-contained.

Part 1. Irregular case

Assume $q(X) \geq 1$. The advantage of the condition $q(X) \geq 1$ lies in that X is equipped with the Albanese map, then we get a natural fibration by doing Stein factorization

$$\begin{tikzcd} X \arrow[d, "f"] \arrow[dr, "\mathrm{alb}_X"] & \\ Y \arrow[r, "\mathrm{alb}_Y"] & \mathrm{Alb}(X) \end{tikzcd} .$$

The problem is reduced to proving subadditivity of Kodaira dimension.

1. Positivity.

Reference:

Sec.4 of a survey: L. Zhang, Birational classification of varieties in positive characteristic.

S. Ejiri, Weak positivity theorem and Frobenius stable canonical rings of geometric generic fibers, J. Algebraic Geom. 26 (2017), no. 4, 691C734.

L. Zhang, Subadditivity of Kodaira dimensions for fibrations of three-folds in positive characteristics, Adv. Math. 354 (2019), 106741, 29.

I suggest reporting Sec. 4 of my survey, which explains the proof of the positivity results from [Ej'17] and [Zh'19]. Ejiri applied Frobenius Iteration, Zhang observed how to use the nefness condition.

2. Subadditivity and abundance.

Reference:

L. Zhang, Abundance for 3-folds with non-trivial Albanese maps in positive characteristic, J. Eur. Math. Soc. (JEMS) 22 (2020), no. 9, 2777C2820.

In fact almost all the techniques are involved when treating the case when X is equipped with a fibration $f : X \to A$ where A is a simple abelian surface. As the argument is too technical, I suggest focus on this case, namely, we need to report

- Sec. 3 which shows how to apply Fourier-Mukai transform to study sheaves on abelian varieties, and
- Sec. 4.2.

Part 2. Regular case

It is not hard to prove abundance for varieties of general type (namely, $\nu(K_X) = 3$, or equivalently K_X is big). When $\nu(K_X) < 3$, only nonvanishing was proved in [XZ19], namely there exists $N > 0$ such that $|NK_X| \neq \emptyset$. XZ follow Miyaoka's strategy, but they need to tackle with the new phenomena: $K_X \cdot c_2(X) < 0$, under this situation they found an foliation structure on X and showed that the nef dimension $n(X) \leq 2$, which concludes abundance easily.

Reference:

Chenyang Xu and Lei Zhang, Nonvanishing for 3-folds in characteristic p ¿ 5, Duke Math. J. 168 (2019), no. 7, 1269C1301.

Sec. 2 treated the new case in char p, the proof is not difficult. Sec. 4 follows the approach from char 0 contributed by Miyaoka and Kawamata, but it uses quite a lot of Langer's results. I suggest admitting the results of fundamental group of Sec. 3.