AN OVERVIEW OF VARIATIONAL ANALYSIS

1. ORIGINS AND MOTIVATIONS

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Introduction

Definition: **variational analysis**

A modern branch of mathematics that extends classical calculus and geometry in the pioneering manner of convex analysis

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Orientation: **shortcomings of traditional mathematics**

Standard analysis was not enough for the optimization problems that got to be important with the invention of computers

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Convexity: **an intermediate development**

The theory of convex sets and functions helped with very new ideas that perhaps could be extended to a much wider domain

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Lecture Goals: **intended participants**

The basic themes of variational analysis in finite dimensions will be explained to researchers who are interested in optimization

Some History

Calculus of variations — a classical subject in mathematics

- optimize curves/surfaces by minimizing integral expressions
- "variations" as tools in deriving optimality conditions
- "variational principles" in physics
- emphasis on theory in function spaces, before computers

Immediate modern descendents:

- optimal control theory (infinite-dimensional) 1960s+
  oriented to engineering of dynamical systems
- linear and nonlinear programming (finite-dimensional) 1950s+
  oriented to operations research and computation

Later developments concerning uncertain information

- stochastic programming: single-stage or multistage
- risk management: in finance and data science

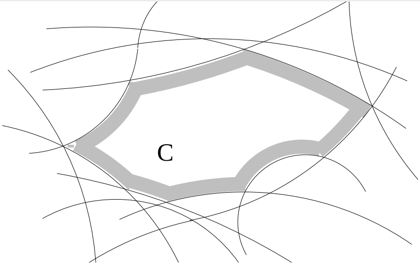
Evolution of Constraints in Mathematical Modeling

Equations — workhorse of classical "descriptive" mathematics

- linear/nonlinear systems of equations, differential equations
- geometric focus: curves, surfaces, differentiable manifolds

Inequalities — essential to modern "prescriptive" mathematics

- systems expressing upper/lower bounds on available choices
- geometric focus: convex sets, nonsmooth boundaries



Convex Sets — Requiring a New Outlook in Geometry

Definition of convexity: of a set $C \subset \Re^n$

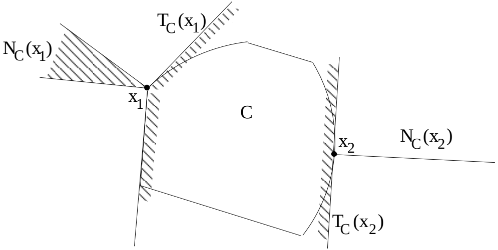
for every pair of points in C, the joining line segment is $\subset C$

Dual characterization: in terms of half-spaces $\{x \mid a{\cdot}x \le b\}$

C is closed convex $\iff$ $C = \bigcap \{\text{collection of closed half-spaces}\}$

interpretation: convexity $\longleftrightarrow$ linear inequality constraints

One-sided geometry of "tangent" and "normal" vectors:

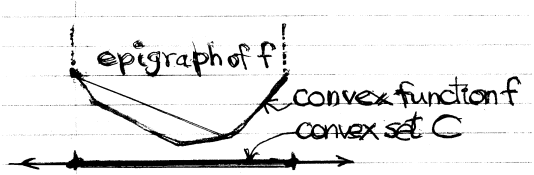


Convex Functions — Needing Epigraphs to Replace Graphs

Definition of convexity: of a function f on a convex set $C \subset \mathbb{R}^n$

$$f((1-\tau)x_0 + \tau x_1) \le (1-\tau)f(x_0) + \tau f(x_1)$$
$$\text{for } x_0, x_1 \in C \text{ and } \tau \in (0,1)$$

Geometric meaning: convexity of the epigraph of f in $\mathbb{R}^n \times \mathbb{R}$



Reformulation: as a convex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, \infty]$

- take $f(x) = \infty$ for $x \notin C$ (the convexity inequality still holds)
- the epigraph is then $\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \ge f(x)\}$
- the set C can be recovered as $\{x \in \mathbb{R}^n \mid f(x) < \infty\}$

the graph of f, now in $\mathbb{R}^n \times \overline{\mathbb{R}}$, is no longer "geometric"

Convex Functions — Subgradients Beyond Gradients

consider $f : \mathbf{R}^n \to (-\infty, \infty]$ and a point x with $f(x) < \infty$

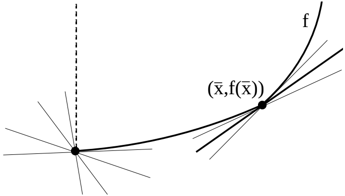
Gradient vectors in classical analysis: for general functions f

$$v = \nabla f(x) \iff f(x') = f(x) + v \cdot (x' - x) + o(|x' - x|)$$

differentiability of f at x, requiring finiteness around x

Subgradient vectors in convex analysis: for convex functions f

$$v \in \partial f(x) \iff f(x') \geq f(x) + v \cdot (x' - x) \text{ for all } x'$$



convex f has its global minimum at $x \iff 0 \in \partial f(x)$

$\partial f(x) = \{v\}$ (singleton) $\iff$ f differentiable at x and $v = \nabla f(x)$

Set-Valued Mappings — Breaking Out of a Mindset

Standard concept of a mapping: M from a set X to a set Y

M assigns to each $x \in X$ one and only one $y \in Y$

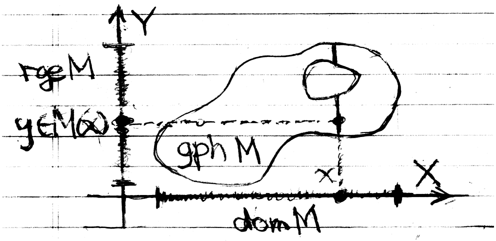
Trouble for convex analysis: what about $\partial f : x \mapsto v \in \partial f(x)$?

∂f can assign to x more than one v, or no v at all

∂f reduces to the gradient mapping ∇f under differentiablity

Framework of set-valued mappings $M : X \rightrightarrows Y$

- the **graph** of M can be **any** subset of $X \times Y$: $\operatorname{gph} M$
- M assigns to x the elements of the set $\{y \mid (x, y) \in \operatorname{gph} M\}$



Evolving Outlook on Problems of Optimization

Nonlinear programming: in terms of functions $f_i : \boldsymbol{R}^n \to \boldsymbol{R}$

minimize $f_0(x)$ under constraints $f_i(x) \leq 0$, or $= 0$, for $i \neq 0$

"feasible" set: $C = \{x \text{ satisfying the constraints}\}$

General perspective: focused on a single function $f : \boldsymbol{R}^n \to \overline{\boldsymbol{R}}$

minimize $f(x)$ over all $x \in \boldsymbol{R}^n$ "feasible" set: $\{x \mid f(x) < \infty\}$

NLP case: $f(x) = f_0(x)$ if $x \in C$, $f(x) = \infty$ otherwise

but there are also many other ways in which f could be specified

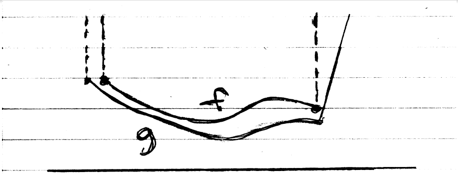
Challenges in this framework: local optimality via $0 \in \partial f(x)$?

- define subgradient mappings ∂f effectively without convexity
- get necessary/sufficient conditions for optimality from them
- develop a "'subdifferential calculus" for handling structure of f

Further "Variations" Needing Attention

What does it mean to "approximate" an optimization problem?

- a problem can be identified with a single function on R^n
- but when are two such functions f and g "close together"?



Key observation: closeness of **epigraphs** is what matters!

the theory of convergence of sets must be brought in!

Shifts in a optimization problem's parameters

minimize $f(p, x)$ in $x \in R^n$ for a function $f : R^d \times R^n \to \overline{R}$

dependence on $p \in R^d$ focuses attention on set-valued mappings

$p \mapsto$ epigraph of $f(p, \cdot)$, $\quad p \mapsto$ set of "solutions"

Lipschitzian Properties Get an Important Role

Lipschitz continuity of a function: f relative to a set X

$\exists\,\kappa$ such that $|f(x') - f(x)| \leq \kappa|x' - x|$ for all $x, x' \in X$

(and more generally for a vector-valued mapping $F : X \to R^m$)

Context: f *continuously differentiable* on a neighborhood of $\bar{x}$

$\Longrightarrow$ f *Lipschitz continuous* on some neighborhood of $\bar{x}$

but Lip. continuity can help even when differentiability is lacking

- Differentiability quantifies infinitesimal **rates** of change
- Lipschitz continuity quantifies **sizes** of possible changes

Extension to solution mappings? maybe set-valued!

$S : p\,(\in R^d) \mapsto x\,(\in R^n)$ solving some p-dependent problem

For a solution $\bar{x}$ corresponding to $\bar{p}$, will there also be solutions x for p near $\bar{p}$, and if so, how might $|x - \bar{x}|$ compare to $|p - \bar{p}|$?

Distinguishing Features of the Resulting Theory

- **One-sided "variations" because of one-sided constraints:**
  inequalities surpass equations in importance
- **Deep understanding of the role of convexity:**
  convex sets and functions, duality
- **Far-reaching generalizations of "calculus":**
  subgradients, subderivatives, variational geometry
- **Extended-real-valued functions, set-valued mappings**
  epigraphs replacing graphs, parametric analysis of solutions
- **Support for, and inspiration from, numerical methodology:**
  optimality conditions, stability analysis, metric regularity
- **Novel approaches to approximation:**
  set limits, variational convergence, graphical convergence

Books in the Background of this Overview

[1] R. T. Rockafellar (1970), *Convex Analysis*, Princeton University Press

[2] R. T. Rockafellar, R. J-B Wets (1998), *Variational Analysis*, Springer-Verlag

[3] A. L. Dontchev, R. T. Rockafellar (2009+2014), *Implicit Functions and Solution Mappings*, Springer-Verlag

In particular: the commentaries in the works provide more history with detailed refences

website: www.math.washington.edu/∼rtr/mypage.html

More Background — Personal

My double location: both in Seattle and on Whidbey Island

my life of mathematics has also been a life of exploring wild nature

Exploring Trails in Ancient Forests

Hiking to Remote Mountain Lakes

Kayaking at my Island Home in October 2021

High Wild Places as Inspiration for Higher Mathematics

but music, literature, travel and of course friendships, too