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# Enhanced Karush–Kuhn–Tucker Conditions for Mathematical Programs with Equilibrium Constraints

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**Abstract** In this paper, we study necessary optimality conditions for nonsmooth mathematical programs with equilibrium constraints. We first show that, unlike the smooth case, the mathematical program with equilibrium constraints linear independent constraint qualification is not a constraint qualification for the strong stationary condition when the objective function is nonsmooth. We then focus on the study of the enhanced version of the Mordukhovich stationary condition, which is a weaker optimality condition than the strong stationary condition. We introduce the quasi-normality and several other new constraint qualifications and show that the enhanced Mordukhovich stationary condition holds under them. Finally, we prove that quasi-normality with regularity implies the existence of a local error bound.

**Keywords** Enhanced Karush–Kuhn–Tucker conditions · Constraint qualification · Mathematical program with equilibrium constraints · Local error bound

## 1 Introduction

In this paper, we study first order necessary optimality conditions for the nonsmooth Mathematical Program with Equilibrium Constraints (MPEC). MPECs form a class of very important problems, since they arise frequently in applications; see [1–3]. MPECs are known to be a difficult class of optimization problems due to the fact that usual constraint qualifications, such as the Linear Independence Constraint Qualification (LICQ) and the Mangasarian–Fromovitz Constraint Qualification (MFCQ), are violated at any feasible point (see [4, Proposition 1.1]). Thus, the classical Karush–

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Kuhn–Tucker (KKT) conditions is not always a necessary optimality condition for a MPEC.

Since there are several different approaches to reformulate MPECs, various stationarity concepts such as Strong, Mordukhovich and Clarke (S, M and C) stationarity arise (see [5–8] for detailed discussions). The S-stationary condition, which is now well known to be equivalent to the classical KKT conditions (see [9]), is the strongest among all stationary concepts for MPECs. For an MPEC with smooth problem data, it is shown that MPEC-LICQ is a constraint qualification for S-stationarity (see [1, 10]). Moreover, MPEC-LICQ is a generic property [11], and hence it is not too stringent and can be satisfied for many smooth MPECs. It is tempting to assume that MPEC-LICQ is also a constraint qualification for MPECs, where the objective function is local Lipschitz but nonsmooth. In this paper, we show through example that MPEC-LICQ is not a constraint qualification for MPECs, where the objective function is nonsmooth.

Recently, Kanzow and Schwartz [12] studied the enhanced KKT conditions for a smooth MPEC. In particular, they introduced the MPEC generalized quasi-normality and pseudo-normality and showed that they are constraint qualifications for the enhanced M-stationary condition. In this paper, we extend Kanzow and Schwartz's results to the nonsmooth case. We show that if the equality functions and the complementarity functions are affine, the inequality function is concave and the abstract constraint set is polyhedral, then the MPEC generalized pseudo-normality holds at each feasible point. In [12], it was shown that the MPEC generalized pseudo-normality is a sufficient condition for the existence of a local error bound for a smooth MPEC. In this paper, we improve this result by showing that the MPEC quasi-normality implies the existence of a local error bound under some reasonable conditions.

Recently, constraint qualifications such as quasi-normality (see [13]), Constant Positive Linear Dependance (CPLD) (see [14]) and Relaxed Constant Positive Linear Dependence (RCPLD) (see [15]) have all been shown to provide weaker constraint qualifications than MFCQ. In this paper we introduce a weaker version of the MPEC-CPLD and show that it is a stronger condition than the MPEC generalized quasi-normality. Consequently this weaker version of the MPEC-CPLD is also a constraint qualification for the enhanced M-stationary condition and a sufficient condition for the existence of a local error bound.

The organization of the paper is as follows. In Sect. 2, we show that MPEC-LICQ is not a constraint qualification if the objective function is nonsmooth. We derive the enhanced M-stationary condition and introduce the corresponding new MPEC constraint qualifications. Finally we prove the error bound results in Sect. 3 and give conclusions in Sect. 4.

## 2 Enhanced Stationary Conditions

The MPEC considered in this paper is formulated as follows:

$$
\begin{aligned}
\text{(MPEC)}\quad & \min_{x\in\mathcal{X}} f(x) \\
& \text{s.t.}\quad h_i(x)=0,\quad i=1,\dots,p,\quad g_j(x)\le 0,\quad j=1,\dots,q, \\
& \qquad G_l(x)\ge 0,\quad H_l(x)\ge 0,\quad G_l(x)H_l(x)=0,\quad \forall l=1,\dots,m,
\end{aligned}
$$

where $f, h_i$ $(i = 1, \ldots, p), g_j$ $(j = 1, \ldots, q) : \mathbb{R}^n \to \mathbb{R}$ are Lipschitz continuous around the point of interest, $G_l, H_l$ $(l = 1, \ldots, m) : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable, and $\mathcal{X}$ is a closed subset of $\mathbb{R}^n$. Let $x^*$ be a feasible point of problem (MPEC). We define the following index sets:

$$\begin{aligned} A(x^*) &:= \{j | g_j(x^*) = 0\}, \\ I_{00} := I_{00}(x^*) &:= \{l | G_l(x^*) = 0,\, H_l(x^*) = 0\}, \\ I_{0+} := I_{0+}(x^*) &:= \{l | G_l(x^*) = 0,\, H_l(x^*) > 0\}, \\ I_{+0} := I_{+0}(x^*) &:= \{l | G_l(x^*) > 0,\, H_l(x^*) = 0\}. \end{aligned}$$

Recall that the MPEC-LICQ holds at a feasible point $x^*$ if the gradient vectors

$$\{\nabla h_i(x^*) | i = 1, \ldots, p\}, \quad \{\nabla g_j(x^*) | j \in A(x^*)\}, \quad \{\nabla G_l(x^*) | l \in I_{00} \cup I_{0+}\}, \\ \{\nabla H_l(x^*) | l \in I_{00} \cup I_{+0}\}$$

are linearly independent (see [11]). The following example shows that MPEC-LICQ may not be a constraint qualification for S-stationary condition if the objective function is not differentiable.

*Example 2.1* Consider the MPEC: $\min -y + |x - y|$ subject to $x \geq 0, y \geq 0, xy = 0$. It is easy to see that $(0, 0)$ is a minimizer and MPEC-LICQ holds at every point of the feasible region. The S-stationary condition is the existence of $\mu \geq 0, \nu \geq 0$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ -1 - \beta \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1}$$

with $\beta \in [-1, 1]$ being an element in the subdifferential of the convex function $|\cdot|$ at the origin. However, (1) never holds and hence $(0, 0)$ is not an S-stationary point.

*Remark 2.1* We may construct a class of MPECs with nonsmooth objectives that have local minimizers satisfying MPEC-LICQ but not S-stationarity. Indeed, consider an MPEC with affine complementarity constraints: $\min f(x)$ s.t. $0 \leq G(x) \perp H(x) \geq 0$, where $a \perp b$ means that the vectors $a$ and $b$ are perpendicular. Since $G(x)$ and $H(x)$ are affine, a local optimal solution to the above MPEC is also a local optimal solution to the penalized problem

$$\min\bigl[f(x) + M\bigl(\|G(x) - y\| + \|H(x) - z\|\bigr)\bigr] \quad \text{s.t.} \quad 0 \leq y \perp z \geq 0$$

for some $M > 0$, where $\|\cdot\|$ denotes the Euclidean norm. For the penalized problem above, MPEC-LICQ holds at each feasible point but the objective function is nonsmooth. However, a local optimal solution is not always an S-stationary point for the penalized problem since otherwise it would also be an S-stationary point for the original problem as well, which may not be true.

We now extend Kanzow and Schwartz's result [12] to the nonsmooth MPEC. In the rest of this paper, we denote by $\partial f(x)$ the limiting subdifferential of function $f$ at $x$ and $\mathcal{N}_{\mathcal{X}}(x)$ the limiting normal cone of set $X$ at $x \in X$. For detailed information on variational analysis, the reader is referred to [16–18].

**Theorem 2.1** *Let $x^*$ be a local minimizer of problem (MPEC). Then, there are multipliers $\alpha, \lambda, \mu, \gamma, \nu$ such that*

(i) $0 \in \alpha\partial f(x^*) + \sum_{i=1}^p \partial(\lambda_i h_i)(x^*) + \sum_{j=1}^q \mu_j \partial g_j(x^*) - \sum_{l=1}^m [\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)] + \mathcal{N}_{\mathcal{X}}(x^*)$;

(ii) $\alpha \geq 0, \mu \geq 0,\ \gamma_l = 0, \forall l \in I_{+0}(x^*),\ \nu_l = 0, \forall l \in I_{0+}(x^*)$, *and either* $\gamma_l > 0$, $\nu_l > 0$ *or* $\gamma_l \nu_l = 0, \forall l \in I_{00}(x^*)$;

(iii) $\alpha, \lambda, \mu, \gamma, \nu$ *are not all equal to zero*;

(iv) *If $\lambda, \mu, \gamma, \nu$ are not all equal to zero, then there exists a sequence $\{x^k\} \subset \mathcal{X}$ converging to $x^*$ such that for all $k$,*

$$f(x^k) < f(x^*),$$
$$\textit{if } \lambda_i \neq 0,\ \textit{then } \lambda_i h_i(x^k) > 0, \quad \textit{if } \mu_j > 0,\ \textit{then } \mu_j g_j(x^k) > 0,$$
$$\textit{if } \gamma_l \neq 0,\ \textit{then } \gamma_l G_l(x^k) < 0, \quad \textit{if } \nu_l \neq 0,\ \textit{then } \nu_l H(x^k) < 0.$$

*Proof* The results can be proved by combining the techniques and the results in [12, Theorem 3.1] and [13, Theorem 1]. □

Based on the result above, we define the following enhanced M-stationary conditions.

**Definition 2.1** (Enhanced M-Stationary Conditions) Let $x^*$ be a feasible point of problem (MPEC). We say the enhanced M-stationary condition holds at $x^*$ iff there are multipliers $\lambda, \mu, \gamma, \nu$ such that

(i) $0 \in \partial f(x^*) + \sum_{i=1}^p \partial(\lambda_i h_i)(x^*) + \sum_{j=1}^q \mu_j \partial g_j(x^*) - \sum_{l=1}^m [\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)] + \mathcal{N}_{\mathcal{X}}(x^*)$;

(ii) $\mu \geq 0$, $\gamma_l = 0$, $\forall l \in I_{+0}(x^*)$, $\nu_l = 0$, $\forall l \in I_{0+}(x^*)$, and either $\gamma_l > 0$, $\nu_l > 0$ or $\gamma_l \nu_l = 0, \forall l \in I_{00}(x^*)$;

(iii) If $\lambda, \mu, \gamma, \nu$ are not all equal to zero, then there exists a sequence $\{x^k\} \subset \mathcal{X}$ converging to $x^*$ such that for all $k$,

$$\text{if } \lambda_i \neq 0,\ \text{then } \lambda_i h_i(x^k) > 0,$$
$$\text{if } \mu_j > 0,\ \text{then } \mu_j g_j(x^k) > 0,$$
$$\text{if } \gamma_l \neq 0,\ \text{then } \gamma_l G_l(x^k) < 0,$$
$$\text{if } \nu_l \neq 0,\ \text{then } \nu_l H(x^k) < 0.$$

We call the multipliers $\lambda, \mu, \gamma, \nu$ the MPEC quasi-normal multipliers corresponding to $x^*$.

Motivated by Theorem 2.1 and the related discussion in [13], we now introduce some MPEC-variant CQs. Note that although Definition 2.2(d) is weaker than the MPEC-CPLD introduced in [19, 20], where all functions involved are continuously differentiable and $\mathcal{X} = \mathbb{R}^n$, for convenience we still refer to it as MPEC-CPLD. The MPEC-RCPLD was first introduced in [21] and has been proven to be a sufficient condition for M-stationarity in [22].

**Definition 2.2** Let $x^*$ be a feasible solution of problem (MPEC).

(a) $x^*$ is said to satisfy MPEC-NNAMCQ iff there is no nonzero vector $(\lambda, \mu, \gamma, \nu)$ such that
 (i) $0 \in \sum_{i=1}^{p} \partial(\lambda_i h_i)(x^*) + \sum_{j=1}^{q} \mu_j \partial g_j(x^*) - \sum_{l=1}^{m} [\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)] + \mathcal{N}_{\mathcal{X}}(x^*)$;
 (ii) $\mu \geq 0$, $\gamma_l = 0$, $\forall l \in I_{+0}(x^*)$; $\nu_l = 0$, $\forall l \in I_{0+}(x^*)$, and either $\gamma_l > 0$, $\nu_l > 0$ or $\gamma_l \nu_l = 0$, $\forall l \in I_{00}(x^*)$.

(b) $x^*$ is said to satisfy MPEC generalized pseudo-normality iff there is no nonzero vector $(\lambda, \mu, \gamma, \nu)$ such that
 (i) $0 \in \sum_{i=1}^{p} \partial(\lambda_i h_i)(x^*) + \sum_{j=1}^{q} \mu_j \partial g_j(x^*) - \sum_{l=1}^{m} [\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)] + \mathcal{N}_{\mathcal{X}}(x^*)$;
 (ii) $\mu \geq 0$, $\gamma_l = 0$, $\forall l \in I_{+0}(x^*)$; $\nu_l = 0$, $\forall l \in I_{0+}(x^*)$, and either $\gamma_l > 0$, $\nu_l > 0$ or $\gamma_l \nu_l = 0$, $\forall l \in I_{00}(x^*)$;
 (iii) There exists a sequence $\{x^k\} \subset \mathcal{X}$ converging to $x^*$ such that for all $k$,

$$\sum_{i=1}^{p} \lambda_i h_i(x^k) + \sum_{j=1}^{q} \mu_j g_j(x^k) - \sum_{l=1}^{m} [\gamma_l G_l(x^k) + \nu_l H_l(x^k)] > 0.$$

(c) $x^*$ is said to satisfy MPEC generalized quasi-normality iff there is no nonzero vector $(\lambda, \mu, \gamma, \nu)$ such that
 (i) $0 \in \sum_{i=1}^{p} \partial(\lambda_i h_i)(x^*) + \sum_{j=1}^{q} \mu_j \partial g_j(x^*) - \sum_{l=1}^{m} [\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)] + \mathcal{N}_{\mathcal{X}}(x^*)$;
 (ii) $\mu \geq 0$, $\gamma_l = 0$, $\forall l \in I_{+0}(x^*)$; $\nu_l = 0$, $\forall l \in I_{0+}(x^*)$, and either $\gamma_l > 0$, $\nu_l > 0$ or $\gamma_l \nu_l = 0$, $\forall l \in I_{00}(x^*)$;
 (iii) If $\lambda, \mu, \gamma, \nu$ are not all equal to zero, then there exists a sequence $\{x^k\} \subset \mathcal{X}$ converging to $x^*$ such that, for all $k$,

$$\begin{aligned}
&\text{if } \lambda_i \neq 0, \ \text{then } \lambda_i h_i(x^k) > 0, \\
&\text{if } \mu_j > 0, \ \text{then } g_j(x^k) > 0, \\
&\text{if } \gamma_l \neq 0, \ \text{then } \gamma_l G_l(x^k) < 0, \\
&\text{if } \nu_l \neq 0, \ \text{then } \nu_l H(x^k) < 0.
\end{aligned}$$

(d) In addition to the basic assumptions for the problem (MPEC), suppose that $h, g$ are continuously differentiable at $x^*$ and $\mathcal{X} = \mathbb{R}^n$. $x^*$ is said to satisfy MPEC-CPLD iff for any indices set $I_0 \subset \mathfrak{P} := \{1, 2, \ldots, p\}$, $J_0 \subset A(x^*)$, $L_0^G \subset$

$I_{0+}(x^*) \cup I_{00}(x^*)$ and $L_0^H \subset I_{+0}(x^*) \cup I_{00}(x^*)$, whenever there exist $\lambda_i$, $\mu_j \geq 0$ for all $j \in J_0$, $\gamma_l$ and $\nu_l$ not all zero, such that

$$0 = \sum_{i \in I_0} \lambda_i \nabla h_i(x^*) + \sum_{j \in J_0} \mu_j \nabla g_j(x^*) - \sum_{l \in L_0^G} \gamma_l \nabla G_l(x^*) - \sum_{l \in L_0^H} \nu_l \nabla H_l(x^*)$$

and either $\gamma_l \nu_l = 0$ or $\gamma_l > 0$, $\nu_l > 0$, $\forall l \in I_{00}(x^*)$, there is a neighborhood $U(x^*)$ of $x^*$ such that, for any $x \in U(x^*)$,

$$\left(\left\{\nabla h_i(x) | i \in I_0\right\}, \left\{\nabla g_j(x) | j \in J_0\right\}, \left\{\nabla G_l(x) | l \in L_0^G\right\}, \left\{\nabla H_l(x) | l \in L_0^H\right\}\right)$$

are linearly dependent.

(e) In addition to the basic assumptions for the problem (MPEC), suppose that $h$, $g$ are continuously differentiable at $x^*$ and $\mathcal{X} = \mathbb{R}^n$. Let $I_0 \subset \mathfrak{P}$ be such that $\{\nabla h_i(x^*)\}_{i \in I_0}$ is a basis for span $\{\nabla h_i(x^*)\}_{i \in \mathfrak{P}}$. $x^*$ is said to satisfy MPEC-RCPLD iff there is a neighborhood $U(x^*)$ of $x^*$ such that
  (i) $\{\nabla h_i(x)\}_{i \in \mathfrak{P}}$ has the same rank for every $x \in U(x^*)$;
  (ii) For every $J_0 \subset A(x^*)$, $L_0^G \subset I_{0+}(x^*) \cup I_{00}(x^*)$ and $L_0^H \subset I_{+0}(x^*) \cup I_{00}(x^*)$, whenever there exist $\lambda_i$, $\mu_j \geq 0$ for all $j \in J_0$, $\gamma_l$ and $\nu_l$ not all zero such that

$$0 = \sum_{i \in I_0} \lambda_i \nabla h_i(x^*) + \sum_{j \in J_0} \mu_j \nabla g_j(x^*) - \sum_{l \in L_0^G} \gamma_l \nabla G_l(x^*) - \sum_{l \in L_0^H} \nu_l \nabla H_l(x^*),$$

and either $\gamma_l \nu_l = 0$ or $\gamma_l > 0$, $\nu_l > 0$, $\forall l \in I_{00}(x^*)$; then the vectors

$$\left(\left\{\nabla h_i(x) | i \in I_0\right\}, \left\{\nabla g_j(x) | j \in J_0\right\}, \left\{\nabla G_l(x) | l \in L_0^G\right\}, \left\{\nabla H_l(x) | l \in L_0^H\right\}\right)$$

are linearly dependent for any $x \in U(x^*)$.

It is easy to see that

$$\begin{aligned} \text{MPEC-NNAMCQ} &\Longrightarrow \text{MPEC generalized pseudo-normality} \\ &\Longrightarrow \text{MPEC generalized quasi-normality.} \end{aligned}$$

For the standard nonsmooth nonlinear program where the equality functions are linear, inequality functions are concave and there is no abstract constraint, Ye and Zhang [13, Proposition 3] showed that the pseudo-normality holds automatically at any feasible point. In what follows, we extend this result to MPEC.

**Theorem 2.2** *Suppose that $h_i$ are linear, $g_j$ are concave, $G_l$, $H_l$ are all linear and $\mathcal{X}$ is polyhedral. Then any feasible point of problem (MPEC) is MPEC generalized pseudo-normal.*

*Proof* We omit the abstract set $\mathcal{X}$ since it can be represented by several linear inequalities. We prove the theorem by contradiction. To the contrary, suppose that there is a feasible point $x^*$ that is not MPEC generalized pseudo-normal. Then there exists nonzero vector $(\lambda, \mu, \gamma, \nu) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ and infeasible sequence $\{x^k\} \subset \mathcal{X}$ converging to $x^*$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{q} \mu_j \partial g_j(x^*) - \sum_{l=1}^{m} [\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)], \tag{2}$$

where $\mu \geq 0, \mu_j = 0, \forall j \notin A(x^*),\ \gamma_l = 0, \forall l \in I_{+0}(x^*),\ \nu_l = 0, \forall l \in I_{0+}(x^*)$ and either $\gamma_l \nu_l = 0$ or $\gamma_l > 0, \nu_l > 0, \forall l \in I_{00}(x^*)$. Furthermore, for each $k$,

$$\sum_{i=1}^{p} \lambda_i h_i(x^k) + \sum_{j=1}^{q} \mu_j g_j(x^k) - \sum_{l=1}^{m} [\gamma_l G_l(x^k) + \nu_l H_l(x^k)] > 0. \tag{3}$$

By the linearity of $h_i, G_l, H_l$ and concavity of $g_j$, we have that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
h_i(x) &= h_i(x^*) + \nabla h_i(x^*)^T (x - x^*), \quad i = 1, \ldots, p, \\
G_l(x) &= G_l(x^*) + \nabla G_l(x^*)^T (x - x^*), \quad l = 1, \ldots, m, \\
H_l(x) &= H_l(x^*) + \nabla H_l(x^*)^T (x - x^*), \quad l = 1, \ldots, m, \\
g_j(x) &\leq g_j(x^*) + \xi_j^T (x - x^*), \quad \forall \xi_j \in \partial g_j(x^*), \quad j = 1, \ldots, q.
\end{aligned}$$

By multiplying these four relations with $\lambda_i, \gamma_l, \nu_l$ and $\mu_j$ and by adding over $i, l$ and $j$ respectively, we obtain that, for all $x \in \mathbb{R}^m$ and all $\xi_j \in \partial g_j(x^*)$, $j = 1, \ldots, q$,

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i h_i(x) + \sum_{j=1}^{q} \mu_j g_j(x) - \sum_{l=1}^{m} (\gamma_l G_l(x) + \nu_l H_l(x)) \\
&\quad \leq \sum_{i=1}^{p} \lambda_i h_i(x^*) + \sum_{j=1}^{q} \mu_j g_j(x^*) - \sum_{l=1}^{m} (\gamma_l G_l(x^*) + \nu_l H_l(x^*)) \\
&\qquad + \left[ \sum_{i=1}^{p} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{q} \mu_j \xi_j - \sum_{l=1}^{m} (\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)) \right]^T (x - x^*) \\
&\quad = \left[ \sum_{i=1}^{p} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{q} \mu_j \xi_j - \sum_{l=1}^{m} (\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)) \right]^T (x - x^*),
\end{aligned}$$

where the last equality holds because we have

$$\lambda_i h_i(x^*) = 0 \quad \text{for all } i \quad \text{and} \quad \sum_{j=1}^{q} \mu_j g_j(x^*) = 0,$$

$$\sum_{l=1}^{m} \gamma_l G_l(x^*) = 0, \quad \sum_{l=1}^{m} \nu_l H_l(x^*) = 0.$$

By (2), there exists $\xi_j^* \in \partial g_j(x^*)$, $j = 1, \dots, q$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{q} \mu_j \xi_j^* - \sum_{l=1}^{m} \big[\gamma_l \nabla G_l(x^*) + \nu_l \nabla H_l(x^*)\big] = 0.$$

Hence it follows that for all $x \in \mathbb{R}^n$, $\sum_{i=1}^{p} \lambda_i h_i(x) + \sum_{j=1}^{q} \mu_j g_j(x) - \sum_{l=1}^{m} [\gamma_l G_l(x) + \nu_l H_l(x)] \le 0$, which contradicts (3). The proof is complete. □

The CPLD was introduced by Qi and Wei in [14] and was used to analyze SQP algorithms. Andreani et al. [23] showed that for smooth nonlinear programs, the CPLD condition implies the quasi-normality and hence is a constraint qualification as well. In what follows, we show that the MPEC-CPLD introduced in this paper implies MPEC generalized quasi-normality. We first recall the following lemma, a proof of which may be found in [15, Lemma 1].

**Lemma 2.1** *If $x = \sum_{i=1}^{m+p} \alpha_i v_i$ with $v_i \in \mathbb{R}^n$ for every $i$, $\{v_i\}_{i=1}^{m}$ is linearly independent and $\alpha_i \neq 0$ for every $i = m+1, \dots, m+p$, then there exist $J \subset \{m+1, \dots, m+p\}$ and scalars $\bar{\alpha}_i$ for every $i \in \{1, \dots, m\} \cup J$ such that*

- $x = \sum_{i \in i,\dots,m \cup J} \bar{\alpha}_i v_i$,
- $\alpha_i \bar{\alpha}_i > 0$ *for every* $i \in J$,
- $\{v_i\}_{i \in \{i,\dots,m\} \cup J}$ *is linearly independent.*

**Theorem 2.3** *Let $x$ be a feasible solution of problem (MPEC) where $h, g$ are continuously differentiable such that MPEC-CPLD holds. Then $x$ is MPEC generalized quasi-normal.*

*Proof* For brevity, we drop the equality and the inequality constraint in the proof, since the main difficulties are induced by the complementarity constraints. Assume that $x$ is feasible and the MPEC-CPLD condition holds at $x$. If $x$ satisfies MPEC-NNAMCQ, we are done. Suppose MPEC-NNAMCQ does not hold. Then, there exists a nonzero vector $(\gamma, \nu) \in \mathbb{R}^m \times \mathbb{R}^m$ such that $0 = -\sum_{l=1}^{m} [\gamma_l \nabla G_l(x) + \nu_l \nabla H_l(x)]$, $\gamma_l = 0, \forall l \in I_{+0}(x)$, $\nu_l = 0, \forall l \in I_{0+}(x)$ and either $\gamma_l \nu_l = 0$ or $\gamma_l > 0$, $\nu_l > 0, \forall l \in I_{00}(x)$. Define the index sets:

$$L_+^G(x) := \big\{l \in I_{0+}(x) | \gamma_l > 0\big\}, \quad L_-^G(x) := \big\{l \in I_{0+}(x) | \gamma_l < 0\big\},$$
$$L_+^H(x) := \big\{l \in I_{+0}(x) | \nu_l > 0\big\}, \quad L_-^H(x) := \big\{l \in I_{+0}(x) | \nu_l < 0\big\},$$
$$I_{00}^{++}(x) := \big\{l \in I_{00}(x) | \gamma_l > 0, \nu_l > 0\big\}, \quad I_{00}^{+0}(x) := \big\{l \in I_{00}(x) | \gamma_l > 0, \nu_l = 0\big\},$$
$$I_{00}^{-0}(x) := \big\{l \in I_{00}(x) | \gamma_l < 0, \nu_l = 0\big\}, \quad I_{00}^{0+}(x) := \big\{l \in I_{00}(x) | \gamma_l = 0, \nu_l > 0\big\},$$
$$I_{00}^{0-}(x) := \big\{l \in I_{00}(x) | \gamma_l = 0, \nu_l < 0\big\}.$$

Since $(\gamma, \nu)$ is a nonzero vector, the union of the above sets must be nonempty and we may write

$$0 = -\Bigg[\sum_{l\in L_+^G(x)} \gamma_l \nabla G_l(x) + \sum_{l\in L_-^G(x)} \gamma_l \nabla G_l(x)\Bigg] - \Bigg[\sum_{l\in L_+^H(x)} \nu_l \nabla H_l(x) + \sum_{l\in L_-^H(x)} \nu_l \nabla H_l(x)\Bigg] - \sum_{l\in I_{00}^{++}(x)} \big[\gamma_l \nabla G_l(x) + \nu_l \nabla H_l(x)\big] - \Bigg[\sum_{l\in I_{00}^{+0}(x)} \gamma_l \nabla G_l(x) + \sum_{l\in I_{00}^{-0}(x)} \gamma_l \nabla G_l(x)\Bigg] - \Bigg[\sum_{l\in I_{00}^{0+}(x)} \nu_l \nabla H_l(x) + \sum_{l\in I_{00}^{0-}(x)} \nu_l \nabla H_l(x)\Bigg].$$

Assume first that $L_+^G(x)$ is nonempty. Let $l_1 \in L_+^G(x)$. Then,

$$-\gamma_{l_1} \nabla G_{l_1}(x) = \Bigg[\sum_{l\in L_+^G(x)\setminus\{l_1\}} \gamma_l \nabla G_l(x) + \sum_{l\in L_-^G(x)} \gamma_l \nabla G_l(x)\Bigg] + \Bigg[\sum_{l\in L_+^H(x)} \nu_l \nabla H_l(x) + \sum_{l\in L_-^H(x)} \nu_l \nabla H_l(x)\Bigg] + \sum_{l\in I_{00}^{++}(x)} \big[\gamma_l \nabla G_l(x) + \nu_l \nabla H_l(x)\big] + \Bigg[\sum_{l\in I_{00}^{+0}(x)} \gamma_l \nabla G_l(x) + \sum_{l\in I_{00}^{-0}(x)} \gamma_l \nabla G_l(x)\Bigg] + \Bigg[\sum_{l\in I_{00}^{0+}(x)} \nu_l \nabla H_l(x) + \sum_{l\in I_{00}^{0-}(x)} \nu_l \nabla H_l(x)\Bigg].$$

If $\nabla G_{i_1}(x) = 0$, the single-element set $\{\nabla G_{i_1}(x)\}$ is linearly dependent. By MPEC-CPLD, the set $\{\nabla G_{i_1}(y)\}$ must be linearly dependent for all $y$ in some neighborhood of $x$. Therefore, $\nabla G_{i_1}(y) = 0$ for all $y$ in an open neighborhood of $x$. Since $G_{i_1}(x) = 0$, this implies that $G_{i_1}(y) = 0$ for all $y$ in that neighborhood. Hence for any sequence $x^k \to x$, $G_{i_1}(x^k) = 0$ always holds. That is, there is no sequence $x^k \to x$ such that $\lambda_{i_1} G_{i_1}(x^k) > 0$.

Assume now that $\nabla G_{i_1}(x) \neq 0$. Then, by Lemma 2.1, there exist index sets

$$\bar{L}_+^G(x) \subset L_+^G(x) \setminus \{i_1\}, \quad \bar{L}_-^G(x) \subset L_-^G(x), \quad \bar{L}_+^H(x) \subset L_+^H(x), \quad \bar{L}_-^H(x) \subset L_-^H(x)$$
$$\bar{I}_{00}^{++}(x) \subset I_{00}^{++}(x), \quad \bar{I}_{00}^{+0}(x) \subset I_{00}^{+0}(x), \quad \bar{I}_{00}^{-0}(x) \subset I_{00}^{-0}(x), \quad \bar{I}_{00}^{0+}(x) \subset I_{00}^{0+}(x),$$
$$\bar{I}_{00}^{0-}(x) \subset I_{00}^{0-}(x)$$

such that the vectors

$$\{\nabla G_l(x)\}_{l\in\bar{L}_+^G(x)},\quad \{\nabla G_l(x)\}_{l\in\bar{L}_-^G(x)},\quad \{\nabla H_l(x)\}_{l\in\bar{L}_+^H(x)},\quad \{\nabla H_l(x)\}_{l\in\bar{L}_-^H(x)},$$
$$\{\nabla G_l(x)\}_{l\in\bar{I}_{00}^{++}(x)},\quad \{\nabla H_l(x)\}_{l\in\bar{I}_{00}^{++}(x)},\quad \{\nabla G_l(x)\}_{l\in\bar{I}_{00}^{+0}(x)},\quad \{\nabla G_l(x)\}_{l\in\bar{I}_{00}^{-0}(x)},$$
$$\{\nabla H_l(x)\}_{l\in\bar{I}_{00}^{0+}(x)},\quad \{\nabla H_l(x)\}_{l\in\bar{I}_{00}^{0-}(x)}$$

are linearly independent and

$$\begin{aligned}-\gamma_{l_1}\nabla G_{l_1}(x) = &\Bigg[\sum_{l\in\bar{L}_+^G(x)}\bar{\gamma}_l\nabla G_l(x) + \sum_{l\in\bar{L}_-^G(x)}\bar{\gamma}_l\nabla G_l(x)\Bigg]\\ &+\Bigg[\sum_{l\in\bar{L}_+^H(x)}\bar{\nu}_l\nabla H_l(x) + \sum_{l\in\bar{L}_-^H(x)}\bar{\nu}_l\nabla H_l(x)\Bigg]\\ &+\sum_{l\in\bar{I}_{00}^{++}(x)}\big[\bar{\gamma}_l\nabla G_l(x) + \bar{\nu}_l\nabla H_l(x)\big]\\ &+\Bigg[\sum_{l\in\bar{I}_{00}^{+0}(x)}\bar{\gamma}_l\nabla G_l(x) + \sum_{l\in\bar{I}_{00}^{-0}(x)}\bar{\gamma}_l\nabla G_l(x)\Bigg]\\ &+\Bigg[\sum_{l\in\bar{I}_{00}^{0+}(x)}\bar{\nu}_l\nabla H_l(x) + \sum_{l\in\bar{I}_{00}^{0-}(x)}\bar{\nu}_l\nabla H_l(x)\Bigg]\end{aligned}$$

with

$$\bar{\gamma}_l>0,\quad \forall l\in\bar{L}_+^G(x),\quad \bar{\gamma}_l<0,\quad \forall l\in\bar{L}_-^G(x),\quad \bar{\nu}_l>0,\quad \forall l\in\bar{L}_+^H(x),$$
$$\nu_l<0,\quad \forall l\in\bar{L}_-^H(x),\quad \bar{\gamma}_l>0,\quad \nu_l>0,\quad \forall l\in\bar{I}_{00}^{++}(x),\quad \bar{\gamma}_l>0,\quad \forall l\in\bar{I}_{00}^{+0}(x),$$
$$\bar{\gamma}_l<0,\quad \forall l\in\bar{I}_{00}^{-0}(x),\quad \bar{\nu}_l>0,\quad \forall l\in\bar{I}_{00}^{0+}(x),\quad \bar{\nu}_l<0,\quad \forall l\in\bar{I}_{00}^{0-}(x).$$

By the linear independence of the vectors and continuity arguments, the vectors

$$\{\nabla G_l(y)\}_{l\in\bar{L}_+^G(x)},\quad \{\nabla G_l(y)\}_{l\in\bar{L}_-^G(x)},\quad \{\nabla H_l(y)\}_{l\in\bar{L}_+^H(x)},\quad \{\nabla H_l(x)\}_{l\in\bar{L}_-^H(x)},$$
$$\{\nabla G_l(y)\}_{l\in\bar{I}_{00}^{++}(x)},\quad \{\nabla H_l(y)\}_{l\in\bar{I}_{00}^{++}(x)},\quad \{\nabla G_l(y)\}_{l\in\bar{I}_{00}^{+0}(x)},\quad \{\nabla G_l(y)\}_{l\in\bar{I}_{00}^{-0}(x)},$$
$$\{\nabla H_l(y)\}_{l\in\bar{I}_{00}^{0+}(x)},\quad \{\nabla H_l(y)\}_{l\in\bar{I}_{00}^{0-}(x)}$$

are linearly independent for all $y$ in a neighborhood of $x$. However, by the MPEC-CPLD assumption, the vectors

$$\gamma_{i_1}\nabla G_{i_1}(y),\quad \{\nabla G_l(y)\}_{l\in\bar{L}_+^G(x)},\quad \{\nabla G_l(y)\}_{l\in\bar{L}_-^G(x)},\quad \{\nabla H_l(y)\}_{l\in\bar{L}_+^H(x)},$$
$$\{\nabla H_l(x)\}_{l\in\bar{L}_-^H(x)},\quad \{\nabla G_l(y)\}_{l\in\bar{I}_{00}^{++}(x)},\quad \{\nabla H_l(y)\}_{l\in\bar{I}_{00}^{++}(x)},\quad \{\nabla G_l(y)\}_{l\in\bar{I}_{00}^{+0}(x)},$$

$$\{\nabla G_l(y)\}_{l\in \bar{I}_{00}^{-0}(x)},\quad \{\nabla H_l(y)\}_{l\in \bar{I}_{00}^{0+}(x)},\quad \{\nabla H_l(y)\}_{l\in \bar{I}_{00}^{0-}(x)}$$

are linearly dependent for all $y$ in a neighborhood of $x$. Therefore, $\lambda_{i_1}\nabla G_{i_1}(y)$ must be a linear combination of the vectors for all $y$ in a neighborhood of $x$.

By [23, Lemma 3.2], there exists a smooth function $\varphi$ defined in a neighborhood of $(0,\dots,0)$ such that, for all $y$ in a neighborhood of $x$,

$$\begin{aligned}-\lambda_{i_1}G_{i_1}(y) = \varphi\big(&\{G_l(y)\}_{l\in\bar{L}_+^G(x)}, \{G_l(y)\}_{l\in\bar{L}_-^G(x)}, \{H_l(y)\}_{l\in\bar{L}_+^H(x)}, \{H_l(y)\}_{l\in\bar{L}_-^H(x)},\\ &\{G_l(y)\}_{l\in\bar{I}_{00}^{++}(x)}, \{H_l(y)\}_{l\in\bar{I}_{00}^{++}(x)}, \{G_l(y)\}_{l\in\bar{I}_{00}^{+0}(x)}, \{G_l(y)\}_{l\in\bar{I}_{00}^{-0}(x)},\\ &\{H_l(y)\}_{l\in\bar{I}_{00}^{0+}(x)}, \{H_l(y)\}_{l\in\bar{I}_{00}^{0-}(x)}\big),\\ \nabla\varphi(0,\dots,0) = \big(&\{\bar{\gamma}\}_{l\in\bar{L}_+^G(x)}, \{\bar{\gamma}\}_{l\in\bar{L}_-^G(x)}, \{\bar{\nu}\}_{l\in\bar{L}_+^H(x)}, \{\bar{\nu}\}_{l\in\bar{L}_-^H(x)}, \{\bar{\gamma}\}_{l\in\bar{I}_{00}^{++}(x)},\\ &\{\bar{\nu}\}_{l\in\bar{I}_{00}^{++}(x)}, \{\bar{\gamma}\}_{l\in\bar{I}_{00}^{+0}(x)}, \{\bar{\gamma}\}_{l\in\bar{I}_{00}^{-0}(x)}, \{\bar{\nu}\}_{l\in\bar{I}_{00}^{0+}(x)}, \{\bar{\nu}\}_{l\in\bar{I}_{00}^{0-}(x)}\big).\end{aligned}$$

Now suppose that $\{x^k\}$ is an infeasible sequence that converges to $x$ and such that

$$\begin{aligned}&G_l(x^k) > 0,\quad \forall l\in\bar{L}_+^G(x),\quad G_l(x^k) < 0,\quad \forall l\in\bar{L}_-^G(x),\\ &H_l(x^k) > 0,\quad \forall l\in\bar{L}_+^H(x),\quad H_l(x^k) < 0,\quad \forall l\in\bar{L}_-^G(x),\\ &G_l(x^k) < 0,\quad H_l(x^k) < 0,\quad \forall l\in\bar{I}_{00}^{++}(x),\\ &G_l(x^k) < 0,\quad \forall l\in\bar{I}_{00}^{+0}(x),\quad G_l(x^k) > 0,\quad \forall l\in\bar{I}_{00}^{-0}(x),\\ &H_l(x^k) < 0,\quad \forall l\in\bar{I}_{00}^{0+}(x),\quad H_l(x^k) > 0,\quad \forall l\in\bar{I}_{00}^{0-}(x).\end{aligned}$$

By virtue of Taylor's expansion of $\varphi$ at $(0,\dots,0)$, for $k$ large enough, we must have $-\lambda_{i_1}G_{i_1}(x^k)\geq 0$. Again, there is no sequence $x^k\to x$ such that $\lambda_{i_1}G_{i_1}(x^k)>0$.

The proofs for the other cases are entirely analogous to the proof for this case. Therefore, MPEC-CPLD implies MPEC generalized quasi-normality. □

The following result follows immediately from Theorem 2.3 and the definitions of the three constraint qualifications.

**Corollary 2.1** *Let $x^*$ be a local minimizer of problem (MPEC). If $x^*$ satisfies MPEC-CPLD, or is MPEC generalized pseudo-normal, or MPEC generalized quasi-normal, then $x^*$ is an enhanced M-stationary point.*

## 3 Error Bound

As one of their main results, Kanzow and Schwartz proved in [12] that the MPEC generalized pseudo-normality implies the existence of a local error bound for smooth MPECs. Combining the proof techniques of [13, Theorem 4] and [12, Theorem 4.5], we can extend [12, Theorem 4.5] to the nonsmooth MPEC. The MPEC generalized quasi-normality is weaker than the MPEC generalized pseudo-normality. It is desirable to find conditions under which the existence of a local error bound holds under the MPEC generalized quasi-normality. We will answer this question in Theorem 3.1.

Before we can do so, we need to prove some preliminary results, which will facilitate the proof of Theorem 3.1.

**Lemma 3.1** *If a feasible point $x^*$ is MPEC generalized quasi-normal, then all feasible points in a neighborhood of $x^*$ are MPEC generalized quasi-normal.*

*Proof* For simplicity, we drop the equality and the inequality constraints in the proof. Assume that the claim is not true. Then we can find a sequence $\{x^k\}$ such that $x^k \neq x^*$ for all $k$, $x^k \to x^*$ and $x^k$ is not quasi-normal for all $k$. This implies, for each $k$, the existence of scalars $\{\gamma^k, \nu^k\}$ not zero and a sequence $\{x^{k,t}\} \subset \mathcal{X}$ such that

(1) $0 \in -\sum_{l=1}^m [\gamma_l^k \nabla G_l(x^k) + \nu_l^k \nabla H_l(x^k)] + \mathcal{N}_{\mathcal{X}}(x^k)$,
(2) $\gamma_l^k = 0, \forall l \in I_{+0}(x^k)$, $\nu_l^k = 0, \forall l \in I_{0+}(x^k)$ and either $\gamma_l^k \nu_l^k = 0$ or $\gamma_l^k > 0$, $\nu_l^k > 0, \forall l \in I_{00}(x^k)$,
(3) $\{x^{k,t}\}$ converges to $x^k$ as $t \to \infty$, and for each $t$, $-\gamma_l^k G_l(x^{k,t}) > 0$, $\forall l \in \mathcal{G}^k$, $-\nu_l^k H_l(x^{k,t}) > 0, \forall l \in \mathcal{H}^k$, where $\mathcal{G}^k = \{l | \gamma_l^k \neq 0\}$ and $\mathcal{H}^k = \{l | \nu_l^k \neq 0\}$.

For each $k$, denote $\tilde{\gamma}^k := \frac{\gamma^k}{\|(\gamma^k, \nu^k)\|}$, $\tilde{\nu}^k := \frac{\nu^k}{\|(\gamma^k, \nu^k)\|}$. Assume, without any loss of generality, that $(\tilde{\gamma}^k, \tilde{\nu}^k) \to (\gamma^*, \nu^*)$. Dividing both sides of (1) above by $\|(\gamma^k, \nu^k)\|$ and taking the limit, we have

(1) $0 \in -\sum_{l=1}^m [\gamma_l^* \nabla G_l(x^*) + \nu_l^* \nabla H_l(x^*)] + \mathcal{N}_{\mathcal{X}}(x^*)$,
(2) $\gamma_l^* = 0, \forall l \in I_{+0}(x^*)$, $\nu_l^* = 0, \forall l \in I_{0+}(x^*)$ and either $\gamma_l^* \nu_l^* = 0$ or $\gamma_l^* > 0$, $\nu_l^* > 0, \forall l \in I_{00}(x^*)$,
(3) $\{\varsigma^k\}$ converges to $x^*$ as $k \to \infty$, and for each $l$, $-\gamma_l^* G_l(\varsigma^k) > 0$, $\forall l \in \mathcal{G}$, $-\nu_l^* H_l(\varsigma^k) > 0, \forall l \in \mathcal{H}$, where $\mathcal{G} = \{l | \gamma_l^* \neq 0\}$, $\mathcal{H} = \{l | \nu_l^* \neq 0\}$.

Indeed, for indices $l \in I_{00}(x^*)$, for each $k$,

$$\begin{array}{rccl} & \tilde{\gamma}_l^k = 0, & \tilde{\nu}_l^k \text{ free}, & \text{if } l \in I_{+0}(x^k), \\ & \tilde{\gamma}_l^k \text{ free}, & \tilde{\nu}_l^k = 0, & \text{if } l \in I_{0+}(x^k), \\ \text{either } \tilde{\gamma}_l^k \tilde{\nu}_l^k = 0 \quad \text{or} & \tilde{\gamma}_l^k > 0, & \tilde{\nu}_l^k > 0, & \text{if } l \in I_{00}(x^k), \end{array}$$

and hence that either $\gamma_l^* \nu_l^* = 0$ or $\gamma_l^* > 0, \nu_l^* > 0, \forall l \in I_{00}(x^*)$. The existence of scalars $\{\gamma^*, \nu^*\}$ and sequence $\{\varsigma^k\}$ violates the MPEC quasi-normality of $x^*$, thus completing the proof. □

Recall that for a closed subset $\Omega \subset \mathbb{R}^n$ and $\bar{x} \in \Omega$, the proximal normal cone to $\Omega$ at $\bar{x}$ and the Fréchet (regular) normal cone to $\Omega$ at $\bar{x}$ are the convex cones

$$\mathcal{N}_\Omega^\pi(\bar{x}) := \left\{\xi \in \mathbb{R}^n | \exists \sigma > 0 \text{ s.t. } \langle \xi, x - \bar{x} \rangle \leq \sigma \|x - \bar{x}\|^2 \quad \forall x \in \Omega\right\},$$

$$\mathcal{N}_\Omega^F(\bar{x}) := \left\{\xi \in \mathbb{R}^n | \limsup_{x \to \bar{x}, x \in \Omega} \frac{\langle \xi, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0\right\},$$

respectively, and $\mathcal{N}_\Omega(\bar{x}) := \limsup_{x\to\bar{x}} \mathcal{N}^F_\Omega(\bar{x}) = \limsup_{x\to\bar{x}} \mathcal{N}^\pi_\Omega(\bar{x})$, where $\limsup_{x\to\bar{x}} \Phi(x)$ denotes the Kuratowski–Painlevé upper (outer) limit. In the following result, we obtain a specific representation of the limiting normal cone to the constraint region in terms of the set of MPEC quasi-normal multipliers.

**Proposition 3.1** *If $\bar{x}$ is MPEC generalized quasi-normal for $\mathcal{C}$, then*

$$\mathcal{N}_\mathcal{C}(\bar{x}) \subset \left\{ \sum_{i=1}^p \partial(\lambda_i h_i)(\bar{x}) + \sum_{j=1}^q \mu_j \partial g_j(\bar{x}) - \sum_{l=1}^m \left[\gamma_l \nabla G_t(\bar{x}) + \nu_l \nabla H_t(\bar{x})\right] + \mathcal{N}_\mathcal{X}(\bar{x}) | (\lambda, \mu, \gamma, \nu) \in M_Q(\bar{x}) \right\},$$

*where $M_Q(\bar{x})$ denotes the set of quasi-normal multipliers corresponding to $\bar{x}$.*

*Proof* For simplicity, we omit the equality and the inequality constraints in the proof. Let $v$ be an element of set $\mathcal{N}_\mathcal{C}(\bar{x})$. By definition, there are sequences $x^l \to \bar{x}$ and $v^l \to v$ with $v^l \in \mathcal{N}^F_\mathcal{C}(x^l)$ and $x^l \in \mathcal{C}$.

Step 1. By Lemma 3.1, for $l$ sufficiently large, $x^l$ is MPEC generalized quasi-normal. By [18, Theorem 6.11], for each $l$, there exists a smooth function $\varphi^l$ that achieves a strict global minimum over $\mathcal{C}$ at $x^l$ with $-\nabla\varphi^l(x^l) = v^l$. Since $x^l$ is a MPEC generalized quasi-normal vector of $\mathcal{C}$, by Theorem 2.1, enhanced M stationary condition holds for problem $\min \varphi^l(x)$ s.t. $x \in \mathcal{C}$. That is, there exists a vector $(\gamma^l, \nu^l)$ such that

$$v^l \in -\sum_{t=1}^m \left[\gamma_t^l \nabla G_t\left(x^l\right) + \nu_t^l \nabla H_t\left(x^l\right)\right] + \mathcal{N}_\mathcal{X}(\bar{x}), \tag{4}$$

with $\gamma_t^l = 0, \forall t \in I_{+0}(x^l)$, $\nu_t^l = 0, \forall l \in I_{0+}(x^l)$ and either $\gamma_t^l \nu_t^l = 0$ or $\gamma_t^l > 0$, $\nu_t^l > 0, \forall l \in I_{00}(x^l)$. Moreover, let $\mathcal{G}^l = \{l | \gamma_t^l \neq 0\}$, $\mathcal{H}^l = \{t | \nu_t^l \neq 0\}$. Then there exists a sequence $\{x^{l,k}\}$ converging to $x^l$ as $k \to \infty$ such that for all $k$, $-\gamma_t^l G_t(x^{l,k}) > 0$, $\forall t \in \mathcal{G}^l$, $-\nu_t^l H_t(x^{l,k}) > 0, \forall t \in \mathcal{H}^l$.

Step 2. We show that the sequence $\{\gamma^l, \nu^l\}$ is bounded. To the contrary, suppose that the sequence $\{\gamma^l, \nu^l\}$ is unbounded. For every $l$, denote $\tilde{\gamma}^l := \frac{\gamma^l}{\|(\gamma^l, \nu^l)\|}$, $\tilde{\nu}^l := \frac{\nu^l}{\|(\gamma^l, \nu^l)\|}$. Assume, without any loss of generality, that $(\tilde{\gamma}^l, \tilde{\nu}^l) \to (\gamma^*, \nu^*)$. Dividing both sides of (4) by $\|(\gamma^l, \nu^l)\|$ and taking the limit; similarly to the proof of Lemma 3.1, we obtain:

(1) $0 \in -\sum_{t=1}^m [\gamma_t^* \nabla G_t(\bar{x}) + \nu_t^* \nabla H_t(\bar{x})] + \mathcal{N}_\mathcal{X}(\bar{x})$,
(2) $\gamma_t^* = 0, \forall t \in I_{+0}(\bar{x})$, $\nu_t^* = 0, \forall t \in I_{0+}(\bar{x})$ and either $\gamma_t^* \nu_t^* = 0$ or $\gamma_t^* > 0$, $\nu_t^* > 0, \forall t \in I_{00}(\bar{x})$,
(3) $\{\varsigma^l\}$ converges to $\bar{x}$ as $l \to \infty$, and for each $l$, $-\gamma_t^* G_t(\varsigma^l) > 0, \forall t \in \mathcal{G}$, $-\nu_t^* H_t(\varsigma^l) > 0, \forall t \in \mathcal{H}$, where $\mathcal{G} = \{t | \gamma_t^* \neq 0\}$, $\mathcal{H} = \{t | \nu_t^* \neq 0\}$.

However, this is impossible since $\bar{x}$ is assumed to be MPEC quasi-normal, and hence the sequence $\{\gamma^l, \nu^l\}$ must be bounded.

Step 3. By virtue of Step 2, without any loss of generality, we assume that $\{\gamma^l, \nu^l\}$ converges to $\{\gamma, \nu\}$ as $l \to \infty$. Taking the limit in (4) as $l \to \infty$, we have

$$v \in -\sum_{l=1}^{m} \big[\gamma_l \nabla G_l(\bar{x}) + \nu_l \nabla H_l(\bar{x})\big] + \mathcal{N}_{\mathcal{X}}(\bar{x})$$

with $\gamma_t = 0, \forall t \in I_{+0}(\bar{x})$, $\nu_t = 0, \forall t \in I_{0+}(\bar{x})$ and either $\gamma_t \nu_t = 0$ or $\gamma_t > 0$, $\nu_t > 0, \forall t \in I_{00}(\bar{x})$. Similarly to Step 2, we can find a subsequence $\{\varsigma^l\}$ that converges to $\bar{x}$ as $l \to \infty$, and for each $l$,

$$-\gamma_t G_t\big(\varsigma^l\big) > 0, \quad \forall t \in \mathcal{G}, \qquad -\nu_t H_t\big(\varsigma^l\big) > 0, \quad \forall t \in \mathcal{H},$$

where $\mathcal{G} = \{t | \gamma_t \neq 0\}$, $\mathcal{H} = \{t | \nu_t \neq 0\}$. □

Taking into account the previous two results, we are now able to obtain a local error bound result for MPECs under the MPEC quasi-normality. For a function $g : \mathbb{R}^n \to \mathbb{R}$, we let $g^+(x) := \max\{0, g(x)\}$ and, if it is vector-valued, then the maximum is taken component-wise.

**Theorem 3.1** *Let $x^* \in \mathcal{C}$, the feasible region of problem (MPEC). Assume that $h_i$ are $C^1$, $g_j(x)$ are subdifferentially regular around $x^*$ in the sense of [17, Definition 1.91(i)] (automatically holds when $g_j$ are convex or $C^1$ around $x^*$), $\mathcal{X}$ is a nonempty, closed and regular in the sense that $\mathcal{N}_{\mathcal{X}}(x) = \mathcal{N}^F_{\mathcal{X}}(x)$ for all $x \in \Omega$. If $x^*$ is MPEC generalized quasi-normal and the strict complementarity condition holds at $x^*$, then there are $\delta, c > 0$ such that*

$$\begin{aligned} &\operatorname{dist}_{\mathcal{C}}(x) \le c\left(\big\|h(x)\big\|_1 + \big\|g^+(x)\big\|_1 + \sum_{l=1}^{m} \operatorname{dist}_{\Omega}\big(G_l(x), H_l(x)\big)\right), \\ &x \in \mathbb{B}\big(x^*, \delta/2\big) \cap \mathcal{X}, \end{aligned} \tag{5}$$

*where $\Omega := \{(a, b) \in \mathbb{R} | a \ge 0, b \ge 0, ab = 0\}$, $\operatorname{dist}_{\Omega}(x)$ is the distance from $x$ to set $\Omega$ and $\|\cdot\|_1$ denotes the $l_1$-norm.*

*Proof* For simplicity, we omit the equality constraints in the proof. By assumption we can find $\delta_0 > 0$ such that $g_j(x)$ are subdifferentially regular for all $x \in \mathbb{B}(x^*, \delta_0)$, the open ball centered at $x^*$ with radius $\delta_0$. Since the required assertion is always true if $x^*$ is in the interior of set $\mathcal{C}$, we only need to consider the case when $x^*$ is in the boundary of $\mathcal{C}$. In this case, (5) can be violated only for $x \notin \mathcal{C}$. Let us take some sequences $\{\not{x}^k\}$ and $\{x^k\}$ such that $\not{x}^k \to x^*$, $\not{x}^k \in \mathcal{X} \setminus \mathcal{C}$, and $x^k = \prod_{\mathcal{C}}(\not{x}^k)$, the projection of $\not{x}^k$ onto the set $\mathcal{C}$. Note that $x^k \to x^*$, since $\|x^k - \not{x}^k\| \le \|\not{x}^k - x^*\|$. For simplicity, we may assume both $\{\not{x}^k\}$ and $\{x^k\}$ belong to $\mathbb{B}(x^*, \delta_0) \cap \mathcal{X}$.

Since $\not{x}^k - x^k \in \mathcal{N}^{\pi}_{\mathcal{C}}(x^k) \subset \mathcal{N}^F_{\mathcal{C}}(x^k)$, we have $\eta^k = \frac{\not{x}^k - x^k}{\|\not{x}^k - x^k\|} \in \mathcal{N}^F_{\mathcal{C}}(x^k)$. Since $x^*$ is quasi-normal, it follows from Lemma 3.1 that the point $x^k$ is also quasi-normal for all sufficiently large $k$ and, without any loss of generality, we may assume that all $x^k$ are quasi-normal. Then, employing Proposition 3.1, there exists a sequence $\{\mu^k, \gamma^k, \nu^k\}$ such that

$$\eta^k \in \sum_{j=1}^{q} \mu_j^k \partial g_j\big(x^k\big) - \sum_{l=1}^{m}\big[\gamma_l^k \nabla G_l\big(x^k\big) + \nu_l^k \nabla H_l\big(x^k\big)\big] + \mathcal{N}_{\mathcal{X}}\big(x^k\big), \tag{6}$$

$\mu^k \geq 0$, $\mu_j^k = 0, \forall j \notin A(x^k)$, $\gamma_l^k = 0, \forall l \in I_{+0}(x^k)$, $\nu_l^k = 0, \forall l \in I_{0+}(x^k)$ and either $\gamma_l^k \nu_l^k = 0$ or $\gamma_l^k > 0, \nu_l^k > 0, \forall l \in I_{00}(x^k)$, and there exists a sequence $\{x^{k,s}\} \subset \mathcal{X}$ such that $x^{k,s} \to x^k$ as $s \to \infty$ and for all $s$, $\mu_j^k g_j(x^{k,s}) > 0$ for $j \in J^k$ and $-\gamma_l^k G_l(x^{k,s}) > 0, \forall l \in \mathcal{G}^k$, $-\nu_l^k H_l(x^{k,s}) > 0, \forall l \in \mathcal{H}^k$, where $J^k = \{j | \mu_j^k > 0\}$ and $\mathcal{G}^k = \{l | \gamma_l^k \neq 0\}$, $\mathcal{H}^k = \{l | \nu_l^k \neq 0\}$. As in the proof of Step 2 in Proposition 3.1, we can show that the quasi-normality of $x^*$ implies that the sequence $\{\mu^k, \gamma^k, \nu^k\}$ is bounded. Therefore, without any loss of generality, we may assume $\{\mu^k, \gamma^k, \nu^k\}$ converges to some vector $\{\mu^*, \gamma^*, \nu^*\}$. Then there exists a number $M_0 > 0$ such that for all $k$, $\|(\mu^k, \gamma^k, \nu^k)\| \leq M_0$. Without any loss of generality, we may assume that $\mathbb{x}^k \in \mathbb{B}(x^*, \frac{\delta_0}{2}) \setminus \mathcal{C}$ and $x^k \in \mathbb{B}(x^*, \delta_0)$ for all $k$. Setting $(\bar{\mu}^k, \bar{\gamma}^k, \bar{\nu}^k) = 2(\mu^k, \gamma^k, \nu^k)$, then from (6), for each $k$, there exist $\rho_j^k \in \partial g_j(x^k), \forall j = 1, \ldots, q$ and $\omega^k \in \mathcal{N}_{\mathcal{X}}(x^k)$ such that

$$\frac{\mathbb{x}^k - x^k}{\|\mathbb{x}^k - x^k\|} = \frac{x^k - \mathbb{x}^k}{\|\mathbb{x}^k - x^k\|} + \sum_{j=1}^{q} \bar{\mu}_j^k \rho_j^k - \sum_{l=1}^{m}\big[\bar{\gamma}^k \nabla G_l\big(x^k\big) + \bar{\nu}_l^k \nabla H_l\big(x^k\big)\big] + \omega^k.$$

We obtain from the discussion above that

$$\begin{aligned}
\|\mathbb{x}^k - x^k\| &= \left\langle \frac{\mathbb{x}^k - x^k}{\|\mathbb{x}^k - x^k\|}, \mathbb{x}^k - x^k \right\rangle \\
&= \left\langle \frac{x^k - \mathbb{x}^k}{\|\mathbb{x}^k - x^k\|}, \mathbb{x}^k - x^k \right\rangle + \sum_{j=1}^{q} \big\langle \bar{\mu}_j^k \rho_j^k, \mathbb{x}^k - x^k \big\rangle \\
&\quad - \sum_{l=1}^{m} \big\langle \bar{\gamma}_l^k \nabla G_l\big(x^k\big) + \bar{\nu}_l^k \nabla H_l\big(x^k\big), \mathbb{x}^k - x^k \big\rangle + \big\langle \omega^k, \mathbb{x}^k - x^k \big\rangle \\
&\leq \sum_{j=1}^{q} \big\langle \bar{\mu}_j^k \rho_j^k, \mathbb{x}^k - x^k \big\rangle - \sum_{l=1}^{m} \big\langle \bar{\gamma}_l^k \nabla G_l\big(x^k\big) + \bar{\nu}_l^k \nabla H_l\big(x^k\big), \mathbb{x}^k - x^k \big\rangle \\
&\quad + o\big(\|\mathbb{x}^k - x^k\|\big) \\
&\leq \sum_{j=1}^{q} \bar{\mu}_j^k \big(g_j\big(\mathbb{x}^k\big) + o\big(\|\mathbb{x}^k - x^k\|\big)\big) - \sum_{l=1}^{m} \bar{\gamma}_l^k \big(G_l\big(\mathbb{x}^k\big) + o\big(\|\mathbb{x}^k - x^k\|\big)\big) \\
&\quad - \sum_{l=1}^{m} \bar{\nu}_l^k \big(H_l\big(\mathbb{x}^k\big) + o\big(\|\mathbb{x}^k - x^k\|\big)\big) + o\big(\|\mathbb{x}^k - x^k\|\big)
\end{aligned}$$

$$\begin{aligned}
&\le 2\Bigg[\sum_{j=1}^{q}\mu_j^k g_j\big(\mathbb{x}^k\big)-\sum_{l=1}^{m}\big(\gamma_l^k G_l\big(\mathbb{x}^k\big)+\nu_l^k H_l\big(\mathbb{x}^k\big)\big)\Bigg] \\
&\qquad + 2\Bigg|\sum_{j=1}^{q}\mu_j^k+\sum_{l=1}^{m}\gamma_l^k+\sum_{l=1}^{m}\nu_l^k+1\Bigg| o\big(\big\|\mathbb{x}^k-x^k\big\|\big) \\
&\le 2\Bigg[\sum_{j=1}^{q}\mu_j^k g_j\big(\mathbb{x}^k\big)-\sum_{l=1}^{m}\big(\gamma_l^k G_l\big(\mathbb{x}^k\big)+\nu_l^k H_l\big(\mathbb{x}^k\big)\big)\Bigg]+\frac{1}{2}\big\|\mathbb{x}^k-x^k\big\|,
\end{aligned}$$

where the first inequality comes from the fact that $\mathcal{X}$ is regular, the second comes from the subdifferential regularity assumption of $g_j(x)$ in $\mathbb{B}(x^*,\delta_0)$, and the last one is valid because, without any loss of generality, we may assume for $k$ sufficiently large, $o(\|\mathbb{x}^k-x^k\|)\le\frac{1}{4(M_0+1)}\|\mathbb{x}^k-x^k\|$ since $\mathbb{x}^k-x^k\to 0$ as $k$ tends to infinity. This means that

$$\operatorname{dist}_{\mathcal{C}}\big(\mathbb{x}^k\big)=\big\|\mathbb{x}^k-x^k\big\|\le 4M_0\Bigg(\sum_{i=1}^{q}g_j^+\big(\mathbb{x}^k\big)+\phi\big(G\big(\mathbb{x}^k\big),H\big(\mathbb{x}^k\big)\big)\Bigg),$$

where

$$\begin{aligned}
&\phi\big(G\big(\mathbb{x}^k\big),H\big(\mathbb{x}^k\big)\big) \\
&\quad=\sum_{l=1}^{m}\max\big\{-G_l\big(\mathbb{x}^k\big),-H_l\big(\mathbb{x}^k\big),G_l\big(\mathbb{x}^k\big)-H_l\big(\mathbb{x}^k\big),\min\big\{G_l\big(\mathbb{x}^k\big),H_l\big(\mathbb{x}^k\big)\big\}\big\}.
\end{aligned}$$

Thus, for any sequence $\{\mathbb{x}^k\}\subset\mathcal{X}$ converging to $x^*$ there exists a number $c>0$ such that

$$\operatorname{dist}_{\mathcal{C}}\big(\mathbb{x}^k\big)\le c\Bigg(\big\|g^+\big(\mathbb{x}^k\big)\big\|_1+\sum_{l=1}^{m}\operatorname{dist}_{\Omega}\big(G_l\big(\mathbb{x}^k\big),H_l\big(\mathbb{x}^k\big)\big)\Bigg),\quad \forall k=1,2,\ldots.$$

This further implies the error bound property at $x^*$. Indeed, suppose the contrary. Then there exists a sequence $\tilde{\mathbb{x}}^k\to x^*$ such that $\tilde{\mathbb{x}}^k\in\mathcal{X}\setminus\mathcal{C}$ and

$$\operatorname{dist}_{\mathcal{C}}\big(\tilde{\mathbb{x}}^k\big)>c\Bigg(\big\|g^+\big(\tilde{\mathbb{x}}^k\big)\big\|_1+\sum_{l=1}^{m}\operatorname{dist}_{\Omega}\big(G_l\big(\tilde{\mathbb{x}}^k\big),H_l\big(\tilde{\mathbb{x}}^k\big)\big)\Bigg)$$

for all $k=1,2,\ldots$, which is a contradiction. □

## 4 Conclusions

We have shown that the MPEC-LICQ is not a constraint qualification for the S-stationary condition if the objective function is not differentiable. Moreover, we have derived the enhanced M-stationary condition and introduced the associated generalized pseudo-normality and quasi-normality conditions for nonsmooth MPECs.

We have also introduced a weaker version of the MPEC-CPLD and shown that it implies the MPEC quasi-normality. Finally we have shown the existence of a local error bound under either the MPEC generalized pseudo-normality or quasi-normality under the subdifferential regularity condition.

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