SIAM J. OPTIM.
Vol. 24, No. 3, pp. 1206–1237

SENSITIVITY ANALYSIS OF THE VALUE FUNCTION FOR PARAMETRIC MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS*

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Abstract. In this paper, we perform sensitivity analysis of the value function for parametric mathematical programs with equilibrium constraints (MPEC). We show that the value function is directionally differentiable in every direction under the MPEC relaxed constant rank regularity condition, the MPEC no nonzero abnormal multiplier constraint qualification, and the restricted inf-compactness condition. This result is new even in the setting of nonlinear programs in which case it means that under the relaxed constant rank regularity condition, the Mangasarian–Fromovitz constraint qualification, and the restricted inf-compactness condition, the value function for parametric nonlinear programs is directionally differentiable in every direction. Enhanced Mordukhovich (M-) and Clarke (C-) stationarity conditions are M- and C-stationarity conditions with certain enhanced properties and the sets of enhanced M- and C-multipliers are usually smaller than their associated sets of M- and C-multipliers. In this paper, we give upper estimates for the subdifferential of the value function in terms of the enhanced M- and C-multipliers, respectively. Such estimates give sharper results than their M- and C-counterparts.

Key words. parametric mathematical program with equilibrium constraints, value function, MPEC relaxed constant rank regularity, sensitivity, directional derivative, subdifferential

AMS subject classifications. 90C26, 90C30, 90C31, 90C33

DOI. 10.1137/130929783

1. Introduction. In practice, it is important to know how well a model responds to perturbations. This paper considers the following mathematical program with equilibrium constraints (MPEC) formulated as a mathematical program with complementarity constraints subject to perturbation p:

$$(\mathrm{MPEC}_p) \qquad \begin{aligned} \min_{x\in\mathcal{C}} \quad & f(x,p) \\ \text{s.t.} \quad & g(x,p)\le 0,\ h(x,p)=0, \\ & 0\le G(x,p)\ \perp\ H(x,p)\ge 0, \end{aligned}$$

where $f:\Re^{n_1+n_2}\to\Re,\ g:\Re^{n_1+n_2}\to\Re^{m_1},\ h:\Re^{n_1+n_2}\to\Re^{m_2}$, and $G,H:\Re^{n_1+n_2}\to\Re^m$, $\mathcal{C}$ is a nonempty and closed subset of $\Re^{n_1}$, and $a\perp b$ means that vector a is perpendicular to vector b. We denote by $\mathcal{X}(p)$ the feasible region of (MPEC$_p$). The value function of (MPEC$_p$) is an extended-valued function defined by

$$\mathcal{V}(p) := \inf\{f(x,p)\mid x\in\mathcal{X}(p)\}$$

*Received by the editors July 19, 2013; accepted for publication (in revised form) May 20, 2014; published electronically August 7, 2014.
http://www.siam.org/journals/siopt/24-3/92978.html
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This author's work was supported by the Innovation Program of Shanghai Municipal Education Commission and a grant from Shanghai University.
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The third author's work was supported in part by NSERC.

and the optimal solution mapping is a set-valued mapping defined by

$$\mathcal{O}(p) := \{x \in \mathcal{X}(p) \mid f(x,p) = \mathcal{V}(p)\}.$$

It is well known that MPECs play very important roles in many fields such as engineering design, economic equilibria, transportation science, and multilevel games. However, these kinds of problems are generally difficult to deal with because their constraints fail to satisfy the standard Mangasarian–Fromovitz constraint qualification (MFCQ) at any feasible point (see, e.g., [34]). A lot of research has been done during the last two decades to study the optimality conditions for MPECs. Such optimality conditions include the Clarke (C-), Mordukhovich (M-), strong (S-), and Bouligand (B-) stationarity conditions; see, e.g., [7, 8, 14, 21, 22, 28, 29, 30, 31, 33, 34]. At the same time, algorithms for solving MPECs have been proposed using a number of approaches such as sequential quadratic programming, penalty function, relaxation, active set identification, etc.; see, e.g., [5, 17, 26] and the references therein. The stability of parametric MPEC has also been studied systematically; see, e.g., [9, 13, 28].

Compared with the developments on optimality conditions, algorithms, and stability, there are only a few publications on the sensitivity of the value function for (MPEC$_p$). In particular, Lucet and Ye [15, 16] addressed the sensitivity of the value function for optimization programs with variational inequality constraints (OPVIC), which includes MPEC as a special case. They established an upper estimate of the limiting subdifferential of the value function in terms of the normal coderivative multipliers for OPVIC. For the case of MPECs, they provided upper estimates for the limiting subdifferentials in terms of various multipliers. Hu and Ralph [11] established formulas for the first- and second-order directional derivatives of the value function under the so-called MPEC linear independence constraint qualification (MPEC-LICQ) by making use of the piecewise programming approach. In a general framework, Mordukhovich and Nam [23] and Mordukhovich, Nam, and Yen [24] derived some results which provide upper estimates for basic and singular subgradients of the value function for general mathematical programs with an abstract set-valued mapping in infinite dimensional Hilbert spaces which include problem (MPEC$_p$) as a subclass.

Janin [12] introduced the constant rank (CR) regularity condition, under which they studied the directional differentiability of the value function for parametric nonlinear programs. Note that the constant rank regularity condition holds automatically if either all the constraint functions are linear or the LICQ holds, but it is not comparable with the MFCQ. Recently, Minchenko and Stakhovski [20] introduced the relaxed constant rank (RCR) regularity condition, which is weaker than the CR regularity condition, and studied the parametric nonlinear programs under the RCR regularity condition. Under the RCR regularity condition, the nonemptiness and uniform compactness of the optimal solution mapping, and the assumption that the constraint functions are $C^{1,1}$ (i.e., the gradient is locally Lipschitzian) with respect to the decision variables, they showed that the value function is directionally differentiable in any direction along which the lower Dini directional derivative of the feasible solution mapping is nonempty at any optimal solution, and they also derived the formula for the first-order directional derivative of the value function [20, Theorem 5].

In this paper, we first extend the results of Minchenko and Stakhovski [20] to MPECs. Even in the case of nonlinear programs, our result improves [20, Theorem 5] in that the constraint functions are only assumed to be C^1 (i.e., the gradient is continous) instead of $C^{1,1}$, and the restricted inf-compactness [3, Hypothesis 6.5.1], which is much weaker than the nonemptiness and uniform compactness of the optimal solution mapping, is required. We establish the formula for the first-order directional derivative in any direction along which the lower Dini directional derivative of the tightened feasible solution mapping of (MPEC$_p$) is nonempty at any optimal solution under the so-called MPEC relaxed constant rank (MPEC-RCR) regularity condition, which is an MPEC version of the RCR regularity condition. We then show that, under the MPEC-RCR condition, the MPEC no nonzero abnormal multiplier constraint qualification (MPEC-NNAMCQ), and the restricted inf-compactness, the value function is directionally differentiable in *every* direction. This result is new even for the special case of nonlinear programs. In this case, since the MPEC-NNAMCQ reduces to the MFCQ, our result means that, under the RCR regularity condition, the MFCQ, and the restricted inf-compactness condition, the value function is directionally differentiable in every direction. Note that, under the MFCQ and the restricted inf-compactness condition, Clarke [3, Corollary 4 on page 243] has given bounds for the upper and lower Dini directional derivatives of the value function. Even under the stronger MPEC-LICQ, our result improves the one given by Hu and Ralph [11] in that the restricted inf-compactness is strictly weaker than the inf-compactness condition. Based on the obtained results, we also study the differentiability of the localized optimal value function of (MPEC$_p$). We show that the localized optimal value function is differentiable under the MPEC-LICQ and the refined second-order sufficient condition (RSOSC), which improve [11, Theorem 1] in that their result requires that the strong second-order sufficient condition holds for all piecewise nonlinear programs of MPECs, which is much stronger than the RSOSC.

For parametric nonlinear programs, under the MFCQ, Gauvin [6] first established the locally Lipschitzian continuity of value functions and also gave an upper estimate for the Clarke subdifferential of value functions in terms of the usual Lagrange multipliers. Since enhanced Fritz John conditions are generally stronger than the classical Fritz John conditions, the set of enhanced multipliers are in general smaller than the set of usual Lagrange multipliers. Recently, Ye and Zhang [32] used the enhanced multipliers to estimate the limiting subdifferential of value functions for nonlinear programs. In this paper, we first obtain the enhanced Fritz John condition for MPECs and then investigate the subdifferentials of the value function for (MPEC$_p$) in terms of enhanced M-multipliers and enhanced C-multipliers, respectively, which provide much sharper upper estimates than those given in [15, 16], in which they gave upper estimates for the limiting subdifferential of the value function in terms of the usual S-, M-, and C-multipliers.

2. Preliminaries. We first give notation that will be used in the paper. We denote by $\mathcal{B}_\delta(x) := \{y \in \Re^n \mid \|y - x\| < \delta\}$ and $\bar{\mathcal{B}}_\delta(x) := \{y \in \Re^n \mid \|y - x\| \leq \delta\}$ the open and the closed ball centered at x with radius $\delta > 0$, respectively. For a point x and a closed set Ω, we denote by $\mathrm{dist}(x, \Omega) := \inf\{\|y - x\| \mid y \in \Omega\}$ the Euclidean distance from x to Ω and by $P_\Omega(x) := \{y \in \Omega \mid \|y - x\| = \mathrm{dist}(x, \Omega)\}$ the set of projection from x to Ω. Moreover, given a set $\Omega \subseteq \Re^n$ and a function $\varphi : \Re^n \to \bar{\Re} := \Re \cup \{\pm\infty\}$, $x^k \to_\Omega x^*$ means that $x^k \to x^*$ with $x^k \in \Omega$ and $x^k \to_\varphi x^*$ means that $x^k \to x^*$ with $\varphi(x^k) \to \varphi(x^*)$.

2.1. Variational analysis. In this subsection, we review some basic concepts and results in variational analysis, which will be used later on. For more details, see, e.g., [2, 3, 21, 22, 27].

Let $\Phi : \Re^n \rightrightarrows \Re^m$ be a set-valued mapping. The *graph* of Φ is defined by

$$\mathrm{gph}\,\Phi := \{(x, y) \mid y \in \Phi(x)\}$$

and the *domain* of Φ is defined by

$$\operatorname{dom}\Phi := \{x \in \Re^n \mid \Phi(x) \neq \emptyset\}.$$

The *Painlevé–Kuratowski outer and inner limit* of Φ with respect to a set Ω at x^* is defined, respectively, by

$$\limsup_{x \to_\Omega x^*} \Phi(x) := \{v \in \Re^m \mid \exists x_k \to_\Omega x^*, \exists v_k \to v \text{ s.t. } v_k \in \Phi(x_k) \text{ for each } k\},$$
$$\liminf_{x \to_\Omega x^*} \Phi(x) := \{v \in \Re^m \mid \forall x_k \to_\Omega x^*, \exists v_k \to v \text{ s.t. } v_k \in \Phi(x_k) \text{ for each } k\}.$$

The *tangent cone* and the *inner tangent cone* of a set Ω at $x^* \in \Omega$ is a closed cone defined, respectively, by

$$\mathcal{T}_\Omega(x^*) := \limsup_{t \downarrow 0} \frac{\Omega - x^*}{t}, \quad \mathcal{T}_\Omega^i(x^*) := \liminf_{t \downarrow 0} \frac{\Omega - x^*}{t}.$$

The tangent cone and inner tangent cone can be equivalently written as (see, e.g., [2, section 2.2.4])

$$\mathcal{T}_\Omega(x^*) = \{d \mid \exists t_k \downarrow 0, \operatorname{dist}(x^* + t_k d, \Omega) = o(t_k)\},$$
$$\mathcal{T}_\Omega^i(x^*) = \{d \mid \operatorname{dist}(x^* + td, \Omega) = o(t) \ \forall t \geq 0\}.$$

Consider a point $x^* \in \Re^n$ with $\varphi(x^*)$ finite. The *regular (or Fréchet) subdifferential* of φ at x^* is defined by

$$\hat{\partial}\varphi(x^*) := \{v \mid \varphi(x) \geq \varphi(x^*) + v^T(x - x^*) + o(\|x - x^*\|)\},$$

the *limiting (or Mordukhovich) subdifferential* of φ at x^* is defined by

$$\partial\varphi(x^*) := \limsup_{x \to_\varphi x^*} \hat{\partial}\varphi(x) = \{v \mid \exists x^k \to_\varphi x^*, \exists v^k \in \hat{\partial}\varphi(x^k) \text{ s.t. } v^k \to v\},$$

and the *horizon (or singular Mordukhovich) subdifferential* of φ at x^* is defined by

$$\partial^\infty\varphi(x^*) := \{v \mid \exists x^k \to_\varphi x^*, \exists v^k \in \hat{\partial}\varphi(x^k),\ \exists t_k \downarrow 0 \text{ s.t. } t_k v^k \to v\}.$$

The following results will be useful.

PROPOSITION 2.1 (see [27, Theorem 9.13]). *Suppose that $\varphi : \Re^n \to \bar{\Re}$ is lower semicontinuous around x^* with $\varphi(x^*)$ finite. Then φ is Lipschitzian around x^* if and only if $\partial^\infty\varphi(x^*) = \emptyset$.*

PROPOSITION 2.2 (see [21, Corollary 8.10]). *If $\varphi : \Re^n \to \bar{\Re}$ is Lipschitzian around $\bar{x}$, then $\partial\varphi(\bar{x}) \neq \emptyset$.*

PROPOSITION 2.3 (see [21, Theorem 3.54]). *Suppose that $\varphi : \Re^n \to \bar{\Re}$ is Lipschitzian around $\bar{x}$. Then φ is strictly differentiable at $\bar{x}$ if and only if $\partial\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$.*

2.2. Directional differentiability for nonlinear programs. In this subsection, we review some related results of directional derivatives of the value function for the parametric nonlinear program (NLP):

$$(\mathrm{NLP}_p) \qquad \min_x \ f(x,p) \quad \text{s.t.} \ g(x,p) \leq 0,\ h(x,p) = 0,$$

where the functions $\{f, g, h\}$ are the same as in the definition of problem (MPEC$_p$). We denote the feasible region of (NLP$_p$) by the set-valued mapping

$$\mathcal{F}(p) := \{x \mid g(x,p) \leq 0,\ h(x,p) = 0\}$$

and define the generalized Lagrangian function of (NLP$_p$) by

$$\mathfrak{L}^r(x,p;\lambda,\mu) := rf(x,p) + g(x,p)^T\lambda + h(x,p)^T\mu, \quad r \geq 0.$$

The value function of (NLP$_p$) is an extended-valued function defined by

$$\mathbb{V}(p) := \inf\{f(x,p) \mid x \in \mathcal{F}(p)\}$$

and the optimal solution mapping is a set-valued mapping defined by

$$\mathfrak{O}(p) := \{x \in \mathcal{F}(p) \mid f(x,p) = \mathbb{V}(p)\}.$$

The set of Lagrange multipliers associated with $x^* \in \mathcal{F}(p^*)$ is

$$\Gamma(x^*,p^*) := \left\{(\lambda,\mu) \mid \nabla_x \mathfrak{L}^1(x^*,p^*;\lambda,\mu) = 0,\ g(x^*,p^*)^T\lambda = 0,\ \lambda \geq 0\right\}.$$

In the following, we introduce some constraint qualifications which are useful in what follows. The CR regularity was first introduced by Janin [12] and its relaxed version–RCR regularity—was first introduced by Minchenko and Stakhovski [20]. Neither of them is comparable with the MFCQ:

(i) The gradients $\{\nabla_x h_i(x^*,p^*) \mid i = 1,\ldots,m_2\}$ are linearly independent;

(ii) there exists d such that

$$\nabla_x g_i(x^*,p^*)^T d < 0, \quad i \in I_g^*,\ \nabla_x h_i(x^*,p^*)^T d = 0, \quad i = 1,\ldots,m_2,$$

where $I_g^* := \{i \mid g_i(x^*,p^*) = 0\}$; see [12].

DEFINITION 2.4. *We say that the set-valued mapping $\mathcal{F}$ is CR regular at $x^* \in \mathcal{F}(p^*)$ if there exists $\delta > 0$ such that, for each $\mathcal{I} \subseteq I_g^*$ and $\mathcal{J} \subseteq \{1,\ldots,m_2\}$, the family of gradients $\{\nabla_x g_i(x,p), \nabla_x h_j(x,p) \mid i \in \mathcal{I}, j \in \mathcal{J}\}$ has the same rank for each $p \in \mathcal{B}_\delta(p^*)$ and $x \in \mathcal{B}_\delta(x^*)$.*

We say that the set-valued mapping $\mathcal{F}$ is RCR regular at $x^* \in \mathcal{F}(p^*)$ if there exists $\delta > 0$ such that, for each $\mathcal{I} \subseteq I_g^*$, the family of gradients $\{\nabla_x g_i(x,p), \nabla_x h_j(x,p) \mid i \in \mathcal{I}, j = 1,\ldots,m_2\}$ has the same rank for each $p \in \mathcal{B}_\delta(p^*)$ and $x \in \mathcal{B}_\delta(x^*)$.

Given a direction d_p, we denote by

$$\mathcal{D}^+\mathbb{V}(p^*;d_p) := \limsup_{t\downarrow 0} t^{-1}(\mathbb{V}(p^*+td_p) - \mathbb{V}(p^*)),$$
$$\mathcal{D}_+\mathbb{V}(p^*;d_p) := \liminf_{t\downarrow 0} t^{-1}(\mathbb{V}(p^*+td_p) - \mathbb{V}(p^*)),$$

the upper and lower Dini directional derivative of $\mathbb{V}$ at p^* in direction d_p, respectively. We define the *directional derivative* of $\mathbb{V}$ at p^* in direction d_p as

$$\mathcal{D}\mathbb{V}(p^*;d_p) := \lim_{t\downarrow 0} t^{-1}(\mathbb{V}(p^*+td_p) - \mathbb{V}(p^*)).$$

Janin [12, Corollary 3.4] obtained the following sufficient condition for the directional differentiabilitiy of the value function under the MFCQ and the CR regularity which are both weaker than the usual regularity condition LICQ.

PROPOSITION 2.5. *Assume that all the functions $\{f, g, h\}$ are C^1 and the optimal solution mapping $\mathfrak{O}(p)$ is nonempty and uniformly compact around p^*, i.e., there exist* *a positive number δ and a bounded set S such that $\emptyset \neq \mathfrak{O}(p)$ for each $p \in \mathcal{B}_\delta(p^*)$ and*

$$\{x \in \mathfrak{O}(p) \mid p \in \mathcal{B}_\delta(p^*)\} \subseteq S.$$

Suppose further that the MFCQ and the CR regularity hold at each $x^* \in \mathfrak{O}(p^*)$. Then the value function $\mathbb{V}$ is directionally differentiable at $p = p^*$ in every direction $d_p \in \Re^{n_2}$ and

$$\mathcal{D}\mathbb{V}(p^*; d_p) = \min_{x^* \in \mathfrak{O}(p^*)} \max_{(\lambda,\mu) \in \Gamma(x^*,p^*)} \nabla_p \mathfrak{L}^1(x^*, p^*; \lambda, \mu)^T d_p.$$

Since MPECs reduce to NLPs when $m = 0$, Theorem 3.11 shows that in Janin's sufficient conditions, the CR regularity can be replaced by the RCR regularity and the nonemptiness and the uniform compactness of the optimal solution mapping can be replaced by a weaker condition called the restricted inf-compactness as defined in Definition 3.8. The following is an example for which Theorem 3.11 can be used to show the directional differentiability of the value function while Proposition 2.5 cannot be used.

Example 2.6. Consider the problem

$$\begin{aligned} \min_x \quad & f(x,p) := (x_1 - 1)^2 + x_2^2 + p \\ \text{s.t.} \quad & h(x,p) := x_1 - 1 = 0, \\ & g(x,p) := x_2 + p \leq 0, \;\; g_2(x,p) := x_1 x_2 \leq 0. \end{aligned}$$

It is not difficult to verify that the optimal solution of the above problem is $(1, \min(-p, 0))$. Thus, the value function is $\mathbb{V}(p) = \min^2(-p, 0) + p$, which is differentiable at $p^* = 0$, and the optimal solution for $p^* = 0$ is $x^* = (1, 0)$. The active index set of inequality constraints at x^* is $I_g^* = \{1, 2\}$. Note that in Definition 2.4, in order to satisfy the CR regularity, taking $\mathcal{J} = \emptyset$, the family of gradients $\{\nabla_x g_i(x,p) | \; i \in I_g^*\}$ must have the same rank for each (x,p) in some neighborhood of (x^*, p^*). But since the family of gradients

$$\{\nabla_x g_i(x,p) | \; i \in I_g^*\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \right\}$$

has rank one at x^* but rank two near x^* when $x_2 \neq 0$, the CR regularity does not hold at x^*. Thus, Proposition 2.5 fails to be applied to this situation. However, since the family of gradients

$$\{\nabla_x h(x,p), \nabla_x g_i(x,p) | \; i \in \mathcal{I}\}$$

has constant rank for each $\mathcal{I} \subseteq I_g^*$ and for all (x,p) in some neighborhood of (x^*, p^*), the RCR regularity holds at x^*. Then, by Theorem 3.11, we can obtain the directional differentiability of the value function.

In the rest of this subsection, we review some sufficient conditions for the directional differentiability of the value function in certain directions under the RCR regularity condition, which are obtained by Minchenko and Stakhovski [20], and then indicate our improvements. The lower Dini directional derivative of set-valued mapping $\mathcal{F}$ at a point $(p^*, x^*) \in \operatorname{gph} \mathcal{F}$ in direction d_p is defined as

$$\begin{aligned} D_+\mathcal{F}(x^*, p^*; d_p) &:= \liminf_{t \downarrow 0} \frac{\mathcal{F}(p^* + td_p) - x^*}{t} \\ &= \{d_x \mid \exists o(t) \text{ s.t. } x^* + td_x + o(t) \in \mathcal{F}(p^* + td_p) \; \forall t \geq 0\}. \end{aligned}$$

For any $(p^*, x^*) \in \mathrm{gph}\, \mathcal{F}$, by the definition, one always has $D_+\mathcal{F}(x^*, p^*; d_p) \subseteq \mathbb{L}_{\mathcal{F}}(x^*, p^*; d_p)$, where $\mathbb{L}_{\mathcal{F}}(x^*, p^*; d_p)$ is the x-projection of the linearization cone of $\mathrm{gph}\, \mathcal{F}$ at (p^*, x^*), i.e.,

$$\mathbb{L}_{\mathcal{F}}(x^*, p^*; d_p) := \left\{ d_x \;\middle|\; \begin{array}{ll} \nabla g_i(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \leq 0, & i \in I_g^* \\ \nabla h_i(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} = 0, & i = 1, \dots, m_2 \end{array} \right\}$$

(see [18, Corollary 6.26]). Moreover, the equality holds provided that the RCR regularity holds as stated in the following lemma.

LEMMA 2.7 (see [20, Corollary 2]). *If the RCR regularity holds at $x^* \in \mathcal{F}(p^*)$ and $D_+\mathcal{F}(x^*, p^*; d_p) \neq \emptyset$, then*

$$D_+\mathcal{F}(x^*, p^*; d_p) = \mathbb{L}_{\mathcal{F}}(x^*, p^*; d_p).$$

Recall that the set-valued mapping $\mathcal{F}(p)$ is calm at $(p^*, x^*) \in \mathrm{gph}\, \mathcal{F}$ if there exist $\delta > 0$ and $\kappa > 0$ such that, for any $p \in \mathcal{B}_\delta(p^*)$ and $x \in \mathcal{B}_\delta(x^*) \cap \mathcal{F}(p)$, there exists a point $\bar{x} \in \mathcal{F}(p^*)$ such that $\|x - \bar{x}\| \leq \kappa \|p - p^*\|$. The following result shows that the RCR regularity condition is slightly stronger than the calmness condition.

LEMMA 2.8 (see [20, Lemma 5]). *If the RCR regularity holds at $x^* \in \mathcal{F}(p^*)$, then there exist $\delta > 0$ and $\kappa > 0$ such that, for any $p \in \mathcal{B}_\delta(p^*)$ and $x \in \mathcal{B}_\delta(x^*) \cap \mathcal{F}(p)$, there exists a point $\bar{x} \in \mathcal{F}(p^*)$ such that $\|x - \bar{x}\| \leq \kappa \|p - p^*\|$ and $g_i(x, p) \leq g_i(\bar{x}, p^*) \leq 0$ for each $i \in I_g^*$.*

Based on Lemmas 2.7 and 2.8, Minchenko and Stakhovski [20, Theorem 5] gave the following result.

PROPOSITION 2.9. *Assume that the objective function f is C^1, the constraint functions $\{g, h\}$ are $C^{1,1}$, and the optimal solution mapping $\mathfrak{O}(p)$ is nonempty and uniformly compact around p^*. Suppose further that $\mathcal{F}$ is RCR regular at each $x^* \in \mathfrak{O}(p^*)$. Then the value function $\mathbb{V}$ is directionally differentiable at $p = p^*$ in each direction $d_p \in \cap_{x^* \in \mathfrak{O}(p^*)} \mathrm{dom}\, D_+\mathcal{F}(x^*, p^*; \cdot)$ and*

$$\begin{aligned} \mathcal{D}\mathbb{V}(p^*; d_p) &= \min_{x^* \in \mathfrak{O}(p^*)} \min_{d_x \in \mathbb{L}_{\mathcal{F}}(x^*, p^*; d_p)} \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \\ &= \min_{x^* \in \mathfrak{O}(p^*)} \max_{(\lambda, \mu) \in \Gamma(x^*, p^*)} \nabla_p \mathfrak{L}^1(x^*, p^*; \lambda, \mu)^T d_p. \end{aligned}$$

Since NLPs can be considered as a special case of MPECs with $m = 0$, Theorem 3.9 shows that, in order to obtain the differentiability of the value function as in Proposition 2.9, the functions $\{f, g, h\}$ are only needed to be C^1 and the nonemptiness and the uniform compactness of the optimal solution mapping can be replaced by the weaker restricted inf-compactness. The following example illustrates this point.

Example 2.10. Consider the problem

$$\begin{aligned} \min_x \quad & f(x, p) := -x^2 + p \\ \text{s.t.} \quad & 0 \leq g_1(x, p) \leq 3, \; g_2(x, p) \leq 0, \end{aligned}$$

where $g_1(x, p) := x + p$ and

$$g_2(x, p) := \begin{cases} \dfrac{2}{3}(x - 3 + p)^{\frac{3}{2}} + x - 3 + p & \text{if } x \geq 3 - p, \\ -\dfrac{2}{3}[-(x - 3) + p]^{\frac{3}{2}} + x - 3 + p & \text{if } x < 3 - p. \end{cases}$$

It is not difficult to see that the feasible region of the above problem is $\{x \mid -p \leq x \leq 3-p\}$, the optimal solution is $x^* = 3$ for $p^* = 0$, and around $p^* = 0$, the value function $\mathbb{V}(p) = -(3-p)^2 + p$, which is differentiable at $p = p^*$. It is easy to verify that the RCR regularity holds at the optimal solution x^*. Moreover, g_2 is C^1 but not $C^{1,1}$ around the optimal solution. Thus, Proposition 2.9 fails to be applied to this situation. However, we can get the directional differentiability of the value function by Theorem 3.9.

3. Directional differentiability of the value function. In this section, we study the directional differentiability of the value function for (MPEC$_p$) under the assumptions that all the involved functions $\{f, g, h, G, H\}$ are C^1 and $\mathcal{C} \equiv \Re^{n_1}$. For a given feasible point $x^* \in \mathcal{X}(p^*)$, we define the following index sets:

$$\begin{cases} I_g^* := \{\, i \mid g_i(x^*, p^*) = 0\}, \\ \mathcal{I}^* := \{\, i \mid G_i(x^*, p^*) = 0 < H_i(x^*, p^*)\}, \\ \mathcal{J}^* := \{\, i \mid G_i(x^*, p^*) = 0 = H_i(x^*, p^*)\}, \\ \mathcal{K}^* := \{\, i \mid G_i(x^*, p^*) > 0 = H_i(x^*, p^*)\}. \end{cases}$$

The *MPEC generalized Lagrangian function* of (MPEC$_p$) is given by

$$\mathscr{L}^r(x, p; \lambda, \mu, u, v) := rf(x,p) + g(x,p)^T\lambda + h(x,p)^T\mu - G(x,p)^T u - H(x,p)^T v, \quad r \geq 0,$$

and the *MPEC linearization cone* at $x^* \in \mathcal{X}(p^*)$ is given by

$$\mathcal{L}(x^*, p^*) := \left\{ d \;\middle|\; \begin{array}{l} \nabla_x g_i(x^*, p^*)^T d \leq 0, \quad i \in I_g^* \\ \nabla_x h_i(x^*, p^*)^T d = 0, \quad i = 1, \ldots, m_2 \\ \nabla_x G_i(x^*, p^*)^T d = 0, \quad i \in \mathcal{I}^* \\ \nabla_x H_i(x^*, p^*)^T d = 0, \quad i \in \mathcal{K}^* \\ 0 \leq \nabla_x G_i(x^*, p^*)^T d \perp \nabla_x H_i(x^*, p^*)^T d \geq 0, \quad i \in \mathcal{J}^* \end{array} \right\}.$$

We say that $x^* \in \mathcal{X}(p^*)$ is *generalized strongly stationary* (generalized S-stationary) to (MPEC$_{p^*}$) if there exist multipliers $(\lambda, \mu, u, v) \in \Re^{m_1} \times \Re^{m_2} \times \Re^m \times \Re^m$ and $r \geq 0$ such that $(r, \lambda, \mu, u, v) \neq 0$ and

$$\begin{cases} \nabla_x \mathscr{L}^r(x^*, p^*; \lambda, \mu, u, v) = 0, \\ \lambda \geq 0, \ g(x^*, p^*)^T \lambda = 0, \\ u_i = 0, \qquad i \in \mathcal{K}^*, \\ v_i = 0, \qquad i \in \mathcal{I}^*, \\ u_i \geq 0, \ v_i \geq 0, \qquad i \in \mathcal{J}^*. \end{cases} \tag{3.1}$$

We define the set of generalized S-multipliers at $x^* \in \mathcal{X}(p^*)$ as

$$\mathcal{M}_S^r(x^*, p^*) := \{(\lambda, \mu, u, v) \mid 0 \neq (r, \lambda, \mu, u, v) \text{ satisfies (3.1)}\}.$$

We say that $x^* \in \mathcal{X}(p^*)$ is a *B-stationary* point or a *piecewise stationary* point of (MPEC$_{p^*}$) if, for each $\mathcal{J} \subseteq \mathcal{J}^*$, there exist multipliers $(\lambda, \mu, u, v) \in \Re^{m_1} \times \Re^{m_2} \times \Re^m \times \Re^m$ such that

$$\begin{cases} \nabla_x \mathscr{L}^1(x^*, p^*; \lambda, \mu, u, v) = 0, \\ \lambda \geq 0, \ g(x^*, p^*)^T \lambda = 0, \\ u_i = 0, \qquad i \in \mathcal{K}^*, \\ v_i = 0, \qquad i \in \mathcal{I}^*, \\ u_i \geq 0, \qquad i \in \mathcal{J}^c, \\ v_i \geq 0, \qquad i \in \mathcal{J}, \end{cases} \tag{3.2}$$

where $\mathcal{J}^c := \mathcal{J}^* \setminus \mathcal{J}$ is the complement of $\mathcal{J}$ in $\mathcal{J}^*$. We denote by $\mathcal{B}_{\mathcal{J}}(x^*, p^*)$ the set of multipliers (λ, μ, u, v) satisfying (3.2) at $x^* \in \mathcal{X}(p^*)$ for each $\mathcal{J} \subseteq \mathcal{J}^*$.

The following constraint qualifications are useful in the subsequent analysis.

DEFINITION 3.1 (see [28]). *We say that MPEC-LICQ holds at $x^* \in \mathcal{X}(p^*)$ if the gradients*

$$\Big\{\nabla_x g_i(x^*, p^*), \nabla_x h_j(x^*, p^*), \nabla_x G_\imath(x^*, p^*), \nabla_x H_\jmath(x^*, p^*) \,\Big|\, i \in I_g^*, j = 1, \dots, m_2, \imath \in \mathcal{I}^* \cup \mathcal{J}^*, \jmath \in \mathcal{K}^* \cup \mathcal{J}^*\Big\}$$

are linearly independent.

The following condition is a parametric version of the MPEC-RCR constraint qualification introduced in [8]. It extends the RCR regularity (see Definition 2.4) to the MPEC setting. It is weaker than the MPEC-LICQ, the MPEC Linear CQ (i.e., all constraint functions are linear with respect to x), and the MPEC constant rank constraint qualification (MPEC-CRCQ) but it is not comparable to the MPEC-NNAMCQ; see [8] for more discussions.

DEFINITION 3.2. *We say that the set-valued mapping $\mathcal{X}$ is MPEC-RCR regular at $x^* \in \mathcal{X}(p^*)$ if there exists $\delta > 0$ such that, for any $\mathcal{I}_1 \subseteq I_g^*$ and $\mathcal{I}_2, \mathcal{I}_3 \subseteq \mathcal{J}^*$, the family of gradients*

$$\Big\{\nabla_x g_i(x, p), \nabla_x h_j(x, p), \nabla_x G_\imath(x, p), \nabla_x H_\jmath(x, p) \,\Big|\, i \in \mathcal{I}_1, j = 1, \dots, m_2, \imath \in \mathcal{I}^* \cup \mathcal{I}_2, \jmath \in \mathcal{K}^* \cup \mathcal{I}_3\Big\}$$

has the same rank for each $p \in \mathcal{B}_\delta(p^*)$ and $x \in \mathcal{B}_\delta(x^*)$.

Let $\mathbb{L}(x^*, p^*; d_p)$ be the x-projection of the MPEC linearization cone of $\operatorname{gph} \mathcal{X}$ at (p^*, x^*), i.e.,

$$\mathbb{L}(x^*, p^*; d_p) := \left\{ d_x \;\middle|\; \begin{array}{l} \nabla g_i(x^*, p^*)^T d \le 0, \quad i \in I_g^* \\ \nabla h_i(x^*, p^*)^T d = 0, \quad i = 1, \dots, m_2 \\ \nabla G_i(x^*, p^*)^T d = 0, \quad i \in \mathcal{I}^* \\ \nabla H_i(x^*, p^*)^T d = 0, \quad i \in \mathcal{K}^* \\ 0 \le \nabla G_i(x^*, p^*)^T d \perp \nabla H_i(x^*, p^*)^T d \ge 0, \quad i \in \mathcal{J}^* \end{array} \right\}.$$

For the sake of simplicity, we denote

$$F(x, p) := \begin{pmatrix} g(x, p) \\ h(x, p) \\ \Psi(x, p) \end{pmatrix}, \quad \Lambda := \Re_-^{m_1} \times \{0\}^{m_2} \times C^m, \tag{3.3}$$

where $\Re_-$ denotes the nonpositive orthant $\{v \in \Re \mid v \le 0\}$ and

$$\Psi(x, p) := \begin{pmatrix} G_1(x, p) \\ H_1(x, p) \\ \vdots \\ G_m(x, p) \\ H_m(x, p) \end{pmatrix}, \quad C := \{\, (a, b) \in \Re^2 \mid 0 \le a \perp b \ge 0\}. \tag{3.4}$$

Thus, the feasible region of (MPEC$_p$) can be rewritten as $\mathcal{X}(p) := \{x \mid F(x, p) \in \Lambda\}$. By direct calculation, we have (see, e.g., [8, 14])

$$\mathbb{L}(x^*, p^*; d_p) = \{d_x \mid \nabla F(x^*, p^*)^T d \in \mathcal{T}_\Lambda(F(x^*, p^*))\}.$$

LEMMA 3.3. *Let $x^* \in \mathcal{X}(p^*)$. Then $D_+\mathcal{X}(x^*, p^*; d_p) \subseteq \mathbb{L}(x^*, p^*; d_p)$. Suppose that the set-valued mapping $\mathcal{X}$ is MPEC R-regular at $x^* \in \mathcal{X}(p^*)$ in direction d_p, i.e., there exist $\kappa > 0$ and $\delta > 0$ such that*

$$\mathrm{dist}(x, \mathcal{X}(p^* + td_p)) \leq \kappa\, \mathrm{dist}(F(x, p^* + td_p), \Lambda) \quad \forall t \in (0, \delta), \forall x \in \mathcal{B}_\delta(x^*).$$

Then

$$D_+\mathcal{X}(x^*, p^*; d_p) = \mathbb{L}(x^*, p^*; d_p).$$

Proof. Let $d_x \in D_+\mathcal{X}(x^*, p^*; d_p)$. By the definition of the lower Dini directional derivative of set-valued mappings, there exists $o(t)$ such that

$$x^* + td_x + o(t) \in \mathcal{X}(p^* + td_p) \quad \forall t \geq 0.$$

Thus, by Taylor's theorem, we have

$$\Lambda \ni F(x^* + td_x + o(t), p^* + td_p) = F(x^*, p^*) + t\nabla F(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} + o(t).$$

Then, it is easy to see that

$$\nabla F(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \in \mathcal{T}_\Lambda(F(x^*, p^*)).$$

Thus, we have $d_x \in \mathbb{L}(x^*, p^*; d_p)$ and hence $D_+\mathcal{X}(x^*, p^*; d_p) \subseteq \mathbb{L}(x^*, p^*; d_p)$.

Next, we show that $\mathbb{L}(x^*, p^*; d_p) \subseteq D_+\mathcal{X}(x^*, p^*; d_p)$. To this end, we first show that

$$\mathcal{T}_\Lambda(F(x^*, p^*)) = \mathcal{T}^i_\Lambda(F(x^*, p^*)).$$

It follows from [14, Lemma 5.3] that

$$\begin{aligned}\mathcal{T}_\Lambda(F(x^*, p^*)) = &\prod_{i=1}^{m_1} \mathcal{T}_{\Re_-}(g_i(x^*, p^*)) \times \prod_{i=1}^{m_2} \mathcal{T}_{\{0\}}(h_i(x^*, p^*)) \\ &\times \prod_{i=1}^{m} \mathcal{T}_C(G_i(x^*, p^*), H_i(x^*, p^*)).\end{aligned}$$

Since $\Re_-$ and $\{0\}$ are convex and closed, it follows from [2, Proposition 2.55] that for each j,

$$\mathcal{T}_{\Re_-}(g_j(x^*, p^*)) = \mathcal{T}^i_{\Re_-}(g_j(x^*, p^*)), \quad \mathcal{T}_{\{0\}}(h_j(x^*, p^*)) = \mathcal{T}^i_{\{0\}}(h_j(x^*, p^*)).$$

Moreover, by direct calculation, it is not hard to get that for each j,

$$\mathcal{T}_C(G_j(x^*, p^*), H_j(x^*, p^*)) = \mathcal{T}^i_C(G_j(x^*, p^*), H_j(x^*, p^*)).$$

Thus, it suffices to show that

$$\begin{aligned}\mathcal{T}^i_\Lambda(F(x^*, p^*)) = &\prod_{j=1}^{m_1} \mathcal{T}^i_{\Re_-}(g_j(x^*, p^*)) \times \prod_{j=1}^{m_2} \mathcal{T}^i_{\{0\}}(h_j(x^*, p^*)) \\ &\times \prod_{j=1}^{m} \mathcal{T}^i_C(G_j(x^*, p^*), H_j(x^*, p^*)).\end{aligned}$$

If $d \in \mathcal{T}^i_\Lambda(F(x^*,p^*))$, then $\mathrm{dist}(F(x^*,p^*)+td,\Lambda)=o(t)$ $(\forall t\geq 0)$. It remains true for any of its components. Thus, we have

$$d \in \prod_{j=1}^{m_1} \mathcal{T}^i_{\Re_-}(g_j(x^*,p^*)) \times \prod_{j=1}^{m_2} \mathcal{T}^i_{\{0\}}(h_j(x^*,p^*)) \times \prod_{j=1}^{m} \mathcal{T}^i_C(G_j(x^*,p^*),H_j(x^*,p^*)).$$

In a similar way, we can show that the converse part also holds. Therefore, we have

$$\mathcal{T}_\Lambda(F(x^*,p^*)) = \mathcal{T}^i_\Lambda(F(x^*,p^*)).$$

Let $d_x \in \mathbb{L}(x^*,p^*;d_p)$. Then, by the definition, we have

$$\nabla F(x^*,p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \in \mathcal{T}_\Lambda(F(x^*,p^*)) = \mathcal{T}^i_\Lambda(F(x^*,p^*)).$$

By the definition of inner tangent cone, we have

$$\mathrm{dist}\left(F(x^*,p^*)+t\nabla F(x^*,p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix},\Lambda\right) = o(t) \quad \forall t\geq 0.$$

Thus, it follows from the MPEC R-regularity in direction d_p and the above equality that

$$\begin{aligned}
\mathrm{dist}(x^*+td_x,\mathcal{X}(p^*+td_p)) &\leq \kappa\,\mathrm{dist}(F(x^*+td_x,p^*+td_p),\Lambda)\\
&= \kappa\,\mathrm{dist}\left(F(x^*,p^*)+t\nabla F(x^*,p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix}+o(t),\Lambda\right)\\
&= \kappa\,\mathrm{dist}\left(F(x^*,p^*)+t\nabla F(x^*,p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix},\Lambda\right)+o(t)\\
&= o(t),
\end{aligned}$$

which implies that $d_x \in D_+\mathcal{X}(x^*,p^*;d_p)$. Thus, we have

$$\mathbb{L}(x^*,p^*;d_p) \subseteq D_+\mathcal{X}(x^*,p^*;d_p),$$

and then $\mathbb{L}(x^*,p^*;d_p) = D_+\mathcal{X}(x^*,p^*;d_p)$. The proof is complete. ∎

Next we investigate some sufficient conditions to ensure that the directional MPEC R-regularity in Lemma 3.3 holds. To this end, for $x^* \in \mathcal{O}(p^*)$, we let

$$\mathcal{X}_T(p) := \left\{ x \;\middle|\; \begin{array}{l} g(x,p)\leq 0,\ h(x,p)=0 \\ G_{\mathcal{I}^*\cup\mathcal{J}^*}(x,p)=0,\ H_{\mathcal{K}^*\cup\mathcal{J}^*}(x,p)=0 \end{array} \right\}.$$

Note that $\mathcal{X}_T(p)$ is the so-called tightened constraint region of problem (MPEC$_p$) [28]. Clearly, $\mathcal{X}_T(p) \subseteq \mathcal{X}(p)$ around (x^*,p^*) and hence $D_+\mathcal{X}_T(x^*,p^*;d_p) \subseteq D_+\mathcal{X}(x^*,p^*;d_p)$ for each d_p. We first give a technical lemma.

LEMMA 3.4. *For any $(a,b)\in\Re^2$, we have*

$$\varphi(a,b) := \min(|a|+\max(-b,0),|b|+\max(-a,0)) \leq \sqrt{2}\mathrm{dist}((a,b),C),$$

where C is the set defined in (3.4).

Proof. To complete the proof, we consider the following five cases:

- $a\geq b\geq 0$: $\varphi(a,b)=b=\mathrm{dist}((a,b),C)$;
- $a>0, b\leq 0$: $\varphi(a,b)=-b=\mathrm{dist}((a,b),C)$;

- $a \le 0, b \le 0$: $\varphi(a,b) = -a - b \le \sqrt{2}\mathrm{dist}((a,b),C)$;
- $a \le 0, b > 0$: $\varphi(a,b) = -a = \mathrm{dist}((a,b),C)$;
- $b > a > 0$: $\varphi(a,b) = a = \mathrm{dist}((a,b),C)$.

In conclusion, we have the desired result. ∎

LEMMA 3.5. *Suppose that the set-valued mapping $\mathcal{X}$ is MPEC-RCR regular at $x^* \in \mathcal{X}(p^*)$. If $D_+\mathcal{X}_T(x^*, p^*; d_p) \neq \emptyset$, then $\mathcal{X}$ is MPEC R-regular at $x^* \in \mathcal{X}(p^*)$ in direction d_p and $D_+\mathcal{X}(x^*, p^*; d_p) = \mathbb{L}(x^*, p^*; d_p) \neq \emptyset$.*

Proof. We will show the desired result by mathematical induction. First we show that the MPEC R-regularity holds when the number m of complementarity constraints is 0. Note that in this case $F(x,p) = (g(x,p), h(x,p))^T$, $\Lambda = \Re^{m_1}_- \times \{0\}^{m_2}$, and $\mathcal{X}(p) = \mathcal{X}_T(p)$. Assume to the contrary that $\mathcal{X}$ is not MPEC R-regular at $x^* \in \mathcal{X}(p^*)$ in direction d_p, i.e., there exist $t_k \downarrow 0$ and $x^k \to x^*$ such that for each k,

$$\mathrm{dist}(x^k, \mathcal{X}(p^k)) > k(\|\max(g(x^k,p^k),0)\| + \|h(x^k,p^k)\|), \tag{3.5}$$

where $p^k := p^* + t_k d_p$. Clearly, $x^k \notin \mathcal{X}(p^k)$. Since $D_+\mathcal{X}(x^*, p^*; d_p) \neq \emptyset$, there exist d_x and $o(t)$ such that $x^* + td_x + o(t) \in \mathcal{X}(p^* + td_p)$ and hence $\mathcal{X}(p^* + td_p) \neq \emptyset$ for $t \ge 0$ sufficiently small. Thus, there exists $\bar{x}^k \in P_{\mathcal{X}(p^k)}(x^k)$ for each k sufficiently large. Since

$$\|x^k - \bar{x}^k\| \le \|x^k - (x^* + t_k d_x + o(t_k))\| \to 0 \text{ as } \quad k \to \infty,$$

we have $\bar{x}^k \to x^*$ as $k \to \infty$. Then it is easy to see that RCR regularity holds at $(\bar{x}^k, p^k)$ for each k sufficiently large. By the definition of the projection, $\bar{x}^k$ is an optimal solution of the following optimization problem for each k sufficiently large:

$$\begin{aligned} \min_x \quad & \|x^k - x\| \\ \text{s.t.} \quad & g(x,p^k) \le 0,\ h(x,p^k) = 0. \end{aligned}$$

Then it follows from [19, Theorem 1] that $\bar{x}^k$ is a KKT point of the above problem, i.e., there exist multipliers (λ^k, μ^k) such that

$$\left\{ \begin{array}{l} \frac{\bar{x}^k - x^k}{\|\bar{x}^k - x^k\|} + \nabla_x g(\bar{x}^k, p^k)\lambda^k + \nabla_x h(\bar{x}^k, p^k)\mu^k = 0, \\ 0 \le \lambda^k \perp -g(\bar{x}^k, p^k) \ge 0. \end{array} \right. \tag{3.6}$$

By Caratheodory's theorem for cone hulls (see, e.g., [1, Proposition 1.3.1]), in a very similar way to the proof of [8, Theorem 4.1], we can show that there exists a bounded multiplier sequence $\{(\bar{\lambda}^k, \bar{\mu}^k)\}$ satisfying (3.6). Without loss of generality, we let $\|(\bar{\lambda}^k, \bar{\mu}^k)\| \le M$ $(M > 0)$ for each k. It follows from (3.6) that for each k sufficiently large,

$$\begin{aligned} \|x^k - \bar{x}^k\| &= \sum_{i=1}^{m_1} \bar{\lambda}_i^k \nabla_x g_i(\bar{x}^k, p^k)^T (x^k - \bar{x}^k) + \sum_{j=1}^{m_2} \bar{\mu}_j^k \nabla_x h_j(\bar{x}^k, p^k)^T (x^k - \bar{x}^k) \\ &= \sum_{i=1}^{m_1} \bar{\lambda}_i^k (g_i(x^k, p^k) - g_i(\bar{x}^k, p^k)) + \sum_{j=1}^{m_2} \bar{\mu}_j^k (h_j(x^k, p^k) - h_j(\bar{x}^k, p^k)) \\ &\quad + o(\|x^k - \bar{x}^k\|) \\ &= \sum_{i=1}^{m_1} \bar{\lambda}_i^k g_i(x^k, p^k) + \sum_{j=1}^{m_2} \bar{\mu}_j^k h_j(x^k, p^k) + o(\|x^k - \bar{x}^k\|) \\ &\le \sum_{i=1}^{m_1} \bar{\lambda}_i^k \max(g_i(x^k, p^k), 0) + \sum_{j=1}^{m_2} |\bar{\mu}_j^k| |h_j(x^k, p^k)| + \frac{1}{2}\|x^k - \bar{x}^k\|. \end{aligned}$$

This means that for each k sufficiently large, we have

$$\mathrm{dist}(x^k, \mathcal{X}(p^k)) = \|x^k - \bar{x}^k\| \le (m_1 + m_2) M(\|\max(g(x^k, p^k), 0)\| + \|h(x^k, p^k)\|),$$

which contradicts (3.5). Thus, $\mathcal{X}$ is MPEC R-regular at $x^* \in \mathcal{X}(p^*)$ in direction d_p when $m = 0$.

Next we show that the MPEC R-regularity holds when $m = 1$. To this end, we consider the following two constraint systems:

$$\mathcal{X}_1(p) := \left\{ x \,\middle|\, \begin{array}{l} g(x,p) \le 0,\ h(x,p) = 0 \\ G_1(x,p) = 0,\ H_1(x,p) \ge 0 \end{array} \right\},$$

$$\mathcal{X}_2(p) := \left\{ x \,\middle|\, \begin{array}{l} g(x,p) \le 0,\ h(x,p) = 0 \\ G_1(x,p) \ge 0,\ H_1(x,p) = 0 \end{array} \right\}.$$

It is easy to verify that $\mathcal{X}_1(p) \cup \mathcal{X}_2(p) = \mathcal{X}(p)$ and $\mathcal{X}_T(p) \subseteq \mathcal{X}_i(p)$ around (x^*, p^*) for $i = 1, 2$. Thus, by the definition of the lower Dini directional derivative of set-valued mappings, we have

$$\emptyset \ne D_+\mathcal{X}_T(x^*, p^*; d_p) \subseteq D_+\mathcal{X}_i(x^*, p^*; d_p) \quad \forall i = 1, 2. \tag{3.7}$$

To complete the proof, we consider the following three cases.

(i) $x^* \in \mathcal{X}_1(p^*) \backslash \mathcal{X}_2(p^*)$: In this case, for any (x, p) sufficiently close to (x^*, p^*),

$$\mathrm{dist}(x, \mathcal{X}(p)) = \mathrm{dist}(x, \mathcal{X}_1(p)).$$

Moreover, we have $0 = G_1(x^*, p^*) < H_1(x^*, p^*)$ which indicates that, for any (x, p) sufficiently close to (x^*, p^*),

$$G_1(x,p) < H_1(x,p) > 0. \tag{3.8}$$

Moreover, the MPEC-RCR regularity assumption implies that $\mathcal{X}_1$ is RCR regular at $x^* \in \mathcal{X}_1(p^*)$. This and (3.7) indicate from the first part of the proof that the MPEC R-regularity holds at $x^* \in \mathcal{X}_1(p^*)$ in direction d_p. Then it follows from (3.8) that there exist $\kappa > 0$ and $\delta > 0$ such that for each $x \in \mathcal{B}_\delta(x^*)$ and $t \in (0, \delta)$,

$$\begin{aligned} \mathrm{dist}(x, \mathcal{X}_1(p^* + td_p) &\le \kappa\big(\|\max(g(x, p^* + td_p), 0)\| + \|h(x, p^* + td_p)\| \\ &\quad + |G_1(x, p^* + td_p)| + \max(-H_1(x, p^* + td_p), 0)\big), \\ &= \kappa\big(\|\max(g(x, p^* + td_p), 0)\| + \|h(x, p^* + td_p)\| \\ &\quad + |G_1(x, p^* + td_p)|\big), \\ &= \kappa\big(\|\max(g(x, p^* + td_p), 0)\| + \|h(x, p^* + td_p)\| \\ &\quad + \mathrm{dist}((G_1(x, p^* + td_p), H_1(x, p^* + td_p)), C)\big). \end{aligned}$$

(ii) $x^* \in \mathcal{X}_2(p^*) \backslash \mathcal{X}_1(p^*)$: In the same way as (i), we can show the desired result in this case.

(iii) $x^* \in \mathcal{X}_1(p^*) \cap \mathcal{X}_2(p^*)$: The MPEC-RCR regularity assumption implies that both $\mathcal{X}_1$ and $\mathcal{X}_2$ are RCR regular at x^*. This and (3.7) imply from the first part of the proof that the MPEC R-regularity holds at $x^* \in \mathcal{X}_1(p^*) \cap \mathcal{X}_2(p^*)$. Thus there exist $\kappa > 0$ and $\delta > 0$ such that for each $x \in \mathcal{B}_\delta(x^*)$ and $t \in (0, \delta)$,

$$\begin{aligned} \mathrm{dist}(x, \mathcal{X}_1(p^* + td_p) &\le \kappa\big(\|\max(g(x, p^* + td_p), 0)\| + \|h(x, p^* + td_p)\| \\ &\quad + |G_1(x, p^* + td_p)| + \max(-H_1(x, p^* + td_p), 0)\big), \\ \mathrm{dist}(x, \mathcal{X}_2(p^* + td_p)) &\le \kappa\big(\|\max(g(x, p^* + td_p), 0)\| + \|h(x, p^* + td_p)\| \\ &\quad + |H_1(x, p^* + td_p)| + \max(-G_1(x, p^* + td_p), 0)\big). \end{aligned}$$

Thus, it follows from the above inequalities and Lemma 3.4 that for any $x \in \mathcal{B}_\delta(x^*)$ and $t \in (0, \delta)$,

$$\begin{aligned}\operatorname{dist}(x, \mathcal{X}(p^* + td_p)) &= \min\big(\operatorname{dist}(x, \mathcal{X}_1(p^* + td_p)), \operatorname{dist}(x, \mathcal{X}_2(p^* + td_p))\big)\\ &\le \sqrt{2}\kappa\big(\|\max(g(x, p^* + td_p), 0)\| + \|h(x, p^* + td_p)\|\\ &\quad + \operatorname{dist}((G_1(x, p^* + td_p), H_1(x, p^* + td_p)), C)\big).\end{aligned}$$

In conclusion, the MPEC R-regularity holds at $x^* \in \mathcal{X}(p^*)$ in direction d_p when $m = 1$. In a similar way, we can show that if the MPEC R-regularity holds when $m \le k$, then the MPEC R-regularity holds when $m = k + 1$. By mathematical induction, we have shown that $\mathcal{X}$ is MPEC R-regular at $x^* \in \mathcal{X}(p^*)$ in direction d_p and hence from Lemma 3.3 we have $D_+\mathcal{X}(x^*, p^*; d_p) = \mathbb{L}(x^*, p^*; d_p)$. The proof is complete. □

Since $\mathcal{X}_T(p)$ coincides with $\mathcal{X}(p)$ around (x^*, p^*) when $\mathcal{J}^* = \emptyset$ (i.e., the strict complementarity holds at $x^* \in \mathcal{X}(p^*)$), we have the following result immediately.

Corollary 3.6. *Suppose that the set-valued mapping $\mathcal{X}$ is MPEC-RCR regular at $x^* \in \mathcal{X}(p^*)$ and the strict complementarity condition holds at $x^* \in \mathcal{X}(p^*)$. If $D_+\mathcal{X}(x^*, p^*; d_p) \neq \emptyset$, then $\mathcal{X}$ is MPEC R-regular at $x^* \in \mathcal{X}(p^*)$ in direction d_p.*

By making use of the piecewise programming approach, we can obtain the following MPEC version of Lemma 2.8.

Lemma 3.7. *Let $x^* \in \mathcal{X}(p^*)$. If $\mathcal{X}$ is MPEC-RCR regular at $x^* \in \mathcal{X}(p^*)$, then there exist $\delta > 0$ and $\kappa > 0$ such that, for any $p \in \mathcal{B}_\delta(p^*)$ and $x \in \mathcal{B}_\delta(x^*) \cap \mathcal{X}(p)$, there exists a point $\bar{x} \in \mathcal{X}(p^*)$ such that*

(a) $\|x - \bar{x}\| \le \kappa\|p - p^*\|$ *and* $g_i(x, p) - g_i(\bar{x}, p^*) \le 0, \quad i \in I_g^*$;

(b) $\big(G_i(x, p) - G_i(\bar{x}, p^*), H_i(x, p) - H_i(\bar{x}, p^*)\big) \in C, \quad i \in \mathcal{J}^*$.

Proof. For each $\mathcal{J} \subseteq \mathcal{J}^*$, consider the piecewise constraint region

$$(3.9)\qquad \mathcal{X}_{\mathcal{J}}(p) := \left\{ x \;\middle|\; \begin{array}{l} g(x, p) \le 0, h(x, p) = 0 \\ G_i(x, p) = 0 \ (i \in \mathcal{I}^* \cup \mathcal{J}),\ G_i(x, p) \ge 0 \ (i \in \mathcal{J}^c) \\ H_i(x, p) = 0 \ (i \in \mathcal{K}^* \cup \mathcal{J}^c),\ H_i(x, p) \ge 0 \ (i \in \mathcal{J}) \end{array} \right\}.$$

Clearly, $x^* \in \mathcal{X}_{\mathcal{J}}(p^*)$ and the RCR regularity holds at $x^* \in \mathcal{X}_{\mathcal{J}}(p^*)$ for each $\mathcal{J} \subseteq \mathcal{J}^*$. Then, for each $\mathcal{J} \subseteq \mathcal{J}^*$, by Lemma 2.8, there exist $\delta_{\mathcal{J}} > 0$ and $\kappa_{\mathcal{J}} > 0$ such that, for any $p \in \mathcal{B}_{\delta_{\mathcal{J}}}(p^*)$ and $x \in \mathcal{B}_{\delta_{\mathcal{J}}}(x^*) \cap \mathcal{X}_{\mathcal{J}}(p)$, there exists a point $\bar{x} \in \mathcal{X}_{\mathcal{J}}(p^*)$ such that

(1) $\|x - \bar{x}\| \le \kappa_{\mathcal{J}}\|p - p^*\|$ and $g_i(x, p) - g_i(\bar{x}, p^*) \le 0,\ i \in I_g^*$;

(2) $G_i(x, p) = G_i(\bar{x}, p^*) = 0$ and $H_i(x, p) \ge H_i(\bar{x}, p^*) \ge 0, \ \ i \in \mathcal{J};\ H_i(x, p) = H_i(\bar{x}, p^*) = 0$ and $G_i(x, p) \ge G_i(\bar{x}, p^*) \ge 0,\ i \in \mathcal{J}^c$.

Since $\mathcal{J}^* = \mathcal{J} \bigcup \mathcal{J}^c$, we have

$$\big(G_i(x, p) - G_i(\bar{x}, p^*), H_i(x, p) - H_i(\bar{x}, p^*)\big) \in C, \quad i \in \mathcal{J}^*.$$

Since it is not hard to verify that $\mathcal{X}(p) = \bigcup_{\mathcal{J} \subseteq \mathcal{J}^*} \mathcal{X}_{\mathcal{J}}(p)$ around (x^*, p^*), we can get the desired result by setting $\kappa := \max_{\mathcal{J} \subseteq \mathcal{J}^*} \kappa_{\mathcal{J}}$ and $\delta := \min_{\mathcal{J} \subseteq \mathcal{J}^*} \delta_{\mathcal{J}}$. □

Motivated by Proposition 2.9, we study the directional derivative of the value function $\mathcal{V}$ under the MPEC-RCR regularity. Note that Theorem 3.9 only requires that the constraint functions are C^1 and the following restricted inf-compactness holds.

Definition 3.8 (see [3, Hypothesis 6.5.1]). *We say that the* restricted inf-compactness *holds around p^* if $\mathcal{V}(p^*)$ is finite and there exist a compact Ω and a* *positive number* ϵ_0 *such that, for all* $p \in \mathcal{B}_{\epsilon_0}(p^*)$ *for which* $\mathcal{V}(p) < \mathcal{V}(p^*) + \epsilon_0$*, the problem* (MPEC$_p$) *has a solution in* Ω*.*

THEOREM 3.9. *Assume that the restricted inf-compactness holds around* p^**. Let* $\mathcal{X}$ *be MPEC-RCR regular at each* $x^* \in \mathcal{O}(p^*)$*. Then the value function* $\mathcal{V}$ *is directionally differentiable at* $p = p^*$ *in every direction* $d_p \in \cap_{x^* \in \mathcal{O}(p^*)} \mathrm{dom}\, D_+\mathcal{X}_T(x^*, p^*; \cdot)$ *and*

$$\begin{aligned}\mathcal{D}\mathcal{V}(p^*; d_p) &= \min_{x^* \in \mathcal{O}(p^*)} \min_{d_x \in \mathbb{L}(x^*, p^*; d_p)} \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \\ &= \min_{x^* \in \mathcal{O}(p^*)} \min_{\mathcal{J} \subseteq \mathcal{J}^*} \max_{(\lambda,\mu,u,v) \in \mathcal{B}_{\mathcal{J}}(x^*,p^*)} \nabla_p \mathscr{L}^1(x^*, p^*; \lambda, \mu, u, v)^T d_p. \end{aligned} \tag{3.10}$$

Proof. If the restricted inf-compactness holds, then it follows from [3, page 246] that $\mathcal{V}(p)$ is lower semicontinuous at p^*. But for the sake of completeness, we give a brief proof here. Let $p^k \to p^*$ as $k \to \infty$ such that

$$\liminf_{p \to p^*} \mathcal{V}(p) = \lim_{k \to \infty} \mathcal{V}(p^k).$$

Assume to the contrary that $\lim_{k \to \infty} \mathcal{V}(p^k) < \mathcal{V}(p^*)$. Then, by the restricted inf-compactness, there exists a bounded sequence $\{x^k\}$ such that $x^k \in \mathcal{O}(p^k)$ for each k sufficiently large, i.e., $\mathcal{V}(p^k) = f(x^k, p^k)$. Without loss of generality, we assume that $x^k \to \bar{x}$ as $k \to \infty$. Then, by the outer semicontinuity of $\mathcal{X}$ (i.e., its graph is closed), we have $\bar{x} \in \mathcal{X}(p^*)$. Thus,

$$\mathcal{V}(p^*) > \lim_{k \to \infty} \mathcal{V}(p^k) = \lim_{k \to \infty} f(x^k, p^k) = f(\bar{x}, p^*) \geq \mathcal{V}(p^*),$$

which gives a contradiction and then $\mathcal{V}$ is lower semicontinuous at p^*. We now show that $\mathcal{V}$ is continuous at p^* in direction $d_p \in \mathrm{dom}\, D_+\mathcal{X}(x^*, p^*; \cdot)$. Since $D_+\mathcal{X}(x^*, p^*; d_p) \neq \emptyset$, there exist $o(t)$ and d_x such that $x^* + td_x + o(t) \in \mathcal{X}(p^* + td_p)$ $\forall t \geq 0$. Thus,

$$\limsup_{t \downarrow 0} \mathcal{V}(p^* + td_p) \leq \limsup_{t \downarrow 0} f(x^* + td_x + o(t), p^* + td_p) = f(x^*, p^*) = \mathcal{V}(p^*),$$

which implies that $\mathcal{V}$ is upper semicontinuous at p^* in direction d_p. Therefore, $\mathcal{V}$ is continuous at p^* in direction $d_p \in \mathrm{dom}\, D_+\mathcal{X}(x^*, p^*; \cdot)$.

Let $x^* \in \mathcal{O}(p^*)$ and $d_p \in \cap_{x^* \in \mathcal{O}(p^*)} \mathrm{dom}\, D_+\mathcal{X}_T(x^*, p^*; \cdot)$. Since $D_+\mathcal{X}_T(x^*, p^*; d_p) \neq \emptyset$, by Lemma 3.5 and the MPEC-RCR regularity assumption,

$$D_+\mathcal{X}(x^*, p^*; d_p) = \mathbb{L}(x^*, p^*; d_p). \tag{3.11}$$

Thus, for each $d_x \in \mathbb{L}(x^*, p^*; d_p)$, by the definition of the lower Dini directional derivative of set-valued mappings, there exists $o(t)$ such that $x^* + td_x + o(t) \in \mathcal{X}(p^* + td_p)$ for $t \geq 0$. Thus, we have

$$\begin{aligned}\mathcal{D}^+\mathcal{V}(p^*; d_p) &= \limsup_{t \downarrow 0} \frac{\mathcal{V}(p^* + td_p) - \mathcal{V}(p^*)}{t} \\ &\leq \limsup_{t \downarrow 0} \frac{f(x^* + td_x + o(t), p^* + td_p) - f(x^*, p^*)}{t} \\ &= \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix}.\end{aligned}$$

Therefore, it is easy to get

$$\mathcal{D}^+\mathcal{V}(p^*; d_p) \leq \min_{x^* \in \mathcal{O}(p^*)} \inf_{d_x \in \mathbb{L}(x^*, p^*; d_p)} \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix}. \tag{3.12}$$

Since $\mathbb{L}(x^*, p^*; d_p) \neq \emptyset$ for each $x^* \in \mathcal{O}(p^*)$, it follows from (3.12) that

$$\mathcal{D}^+\mathcal{V}(p^*; d_p) < \infty. \tag{3.13}$$

On the other hand, let $\{t_k\}$ be a positive sequence converging to 0 such that

$$\mathcal{D}_+\mathcal{V}(p^*; d_p) = \lim_{k\to\infty} \frac{\mathcal{V}(p^* + t_k d_p) - \mathcal{V}(p^*)}{t_k}.$$

It follows from (3.13) that $\mathcal{D}_+\mathcal{V}(p^*; d_p) < \infty$. Thus, it is easy to see that there exists $k_0 > 0$ such that

$$\mathcal{V}(p^* + t_k d_p) - \mathcal{V}(p^*) < \epsilon_0 \quad \forall k \geq k_0,$$

where ϵ_0 is defined as in Definition 3.8. Let $p^k := p^* + t_k d_p$. By the restricted inf-compactness assumption, there exists $x^k \in \mathcal{O}(p^k) \subseteq \mathcal{X}(p^k)$ for any k sufficiently large such that $\{x^k\}$ is bounded. Without loss of generality, we assume that $x^k \to \bar{x}$. Obviously, $\bar{x} \in \mathcal{X}(p^*)$ by the outer semicontinuity of $\mathcal{X}$. Since

$$\mathcal{V}(p^k) = \mathcal{V}(p^*) + t_k \mathcal{D}_+\mathcal{V}(p^*, d_p) + o(t_k),$$

it follows that

$$f(\bar{x}, p^*) = \lim_{k\to\infty} f(x^k, p^k) = \limsup_{k\to\infty} \mathcal{V}(p^k) = \mathcal{V}(p^*),$$

which implies

$$\bar{x} \in \mathcal{O}(p^*). \tag{3.14}$$

Since the MPEC-RCR regularity holds at $\bar{x} \in \mathcal{O}(p^*)$, it follows from Lemma 3.7 that for each sufficiently large k, there exist $\kappa > 0$ independent of k and a sequence $\{x^{k'} \in \mathcal{X}(p^*)\}$ such that

(a) $\|x^k - x^{k'}\| \leq \kappa \|p^k - p^*\|$, $g_i(x^k, p^k) - g_i(x^{k'}, p^*) \leq 0$ $(i \in \bar{I}_g)$;

(b) $\big(G_i(x^k, p^k) - G_i(x^{k'}, p^*), H_i(x^k, p^k) - H_i(x^{k'}, p^*)\big) \in C$ $(i \in \bar{\mathcal{J}})$,

where $\bar{I}_g := \{\, i \mid g_i(\bar{x}, p^*) = 0\}$ and $\bar{\mathcal{J}} := \{\, i \mid G_i(\bar{x}, p^*) = 0 = H_i(\bar{x}, p^*)\}$. It follows that for each k sufficiently large,

$$\begin{gathered} g_i(x^k, p^k) - g_i(x^{k'}, p^*) \leq 0 \;\; (i \in \bar{I}_g), \quad h_i(x^k, p^k) - h_i(x^{k'}, p^*) = 0 \;\; (i = 1, \ldots, m_2), \\ G_i(x^k, p^k) - G_i(x^{k'}, p^*) = 0 \;\; (i \in \bar{\mathcal{I}}), \quad H_i(x^k, p^k) - H_i(x^{k'}, p^*) = 0 \;\; (i \in \bar{\mathcal{K}}), \\ \big(G_i(x^k, p^k) - G_i(x^{k'}, p^*), H_i(x^k, p^k) - H_i(x^{k'}, p^*)\big) \in C \;\; (i \in \bar{\mathcal{J}}), \end{gathered}$$

where $\bar{\mathcal{I}} := \{\, i \mid G_i(\bar{x}, p^*) = 0 < H_i(\bar{x}, p^*)\}$ and $\bar{\mathcal{K}} := \{\, i \mid G_i(\bar{x}, p^*) > 0 = H_i(\bar{x}, p^*)\}$. Since it is easy to see that $\{\frac{x^k - x^{k'}}{t_k}\}$ is bounded, we assume without loss of generality that $\frac{x^k - x^{k'}}{t_k} \to x'$ and then $x^k = x^{k'} + t_k x' + o_1(t_k)$, where $\frac{o_1(t)}{t} \to 0$ as $t \to 0$. Thus, we have

$$\begin{gathered} \nabla g_i(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix} \leq 0 \;\; (i \in \bar{I}_g), \quad \nabla h_i(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix} = 0 \;\; (i = 1, \ldots, m_2), \\ \nabla G_i(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix} = 0 \;\; (i \in \bar{\mathcal{I}}), \quad \nabla H_i(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix} = 0 \;\; (i \in \bar{\mathcal{K}}), \\ \left(\nabla_x G_i(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix}, \nabla_x H_i(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix}\right) \in C, \end{gathered}$$

which indicates that

$$x' \in \mathbb{L}(\bar{x}, p^*; d_p). \tag{3.15}$$

Moreover, since $x^{k'} \in \mathcal{X}(p^*)$, we have

$$\begin{aligned}
\mathcal{D}_+\mathcal{V}(p^*; d_p) &= \lim_{k\to\infty} \frac{\mathcal{V}(p^k) - \mathcal{V}(p^*)}{t_k} \\
&\geq \lim_{k\to\infty} \frac{f(x^k, p^k) - f(x^{k'}, p^*)}{t_k} \\
&= \lim_{k\to\infty} \frac{f(x^{k'} + t_k x' + o_1(t_k), p^k) - f(x^{k'}, p^*)}{t_k} \\
&= \nabla f(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix}.
\end{aligned}$$

Therefore, it follows from (3.14) and (3.15) that

$$\mathcal{D}_+\mathcal{V}(p^*; d_p) \geq \nabla f(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix} \geq \min_{\bar{x}\in\mathcal{O}(p^*)} \inf_{x'\in\mathbb{L}(\bar{x}, p^*; d_p)} \nabla f(\bar{x}, p^*)^T \begin{pmatrix} x' \\ d_p \end{pmatrix}.$$

This and (3.12) indicate that

$$\mathcal{D}\mathcal{V}(p^*; d_p) = \min_{x^*\in\mathcal{O}(p^*)} \inf_{d_x\in\mathbb{L}(x^*, p^*; d_p)} \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix}. \tag{3.16}$$

Next we consider the directional derivative of value function from the dual perspective. For each $x^* \in \mathcal{O}(p^*)$, we consider the piecewise feasible region $\mathcal{X}_{\mathcal{J}}$ defined as in (3.9) associated with $x^* \in \mathcal{O}(p^*)$ and let $\mathbb{L}_{\mathcal{J}}(x^*, p^*; d_p)$ be the x-projection of the linearization cone of ghp $\mathcal{X}_{\mathcal{J}}$ at (p^*, x^*). It is not hard to verify that

$$\mathbb{L}(x^*, p^*; d_p) = \bigcup_{\mathcal{J}\subseteq\mathcal{J}^*} \mathbb{L}_{\mathcal{J}}(x^*, p^*; d_p). \tag{3.17}$$

Since $\mathcal{X}(p) = \bigcup_{\mathcal{J}\subseteq\mathcal{J}^*} \mathcal{X}_{\mathcal{J}}(p)$ around (x^*, p^*), x^* is an optimal solution for each piecewise problem. Moreover, by the MPEC-RCR regularity assumption, the RCR regularity holds at $x^* \in \mathcal{X}_{\mathcal{J}}(p^*)$ and hence $\mathcal{B}_{\mathcal{J}}(x^*, p^*) \neq \emptyset$ for each $\mathcal{J} \subseteq \mathcal{J}^*$ (see, e.g., [19, Theorem 1]). It follows from (3.17) and the duality theorem in linear programming that for each $x^* \in \mathcal{O}(p^*)$,

$$\begin{aligned}
&\inf_{d_x\in\mathbb{L}(x^*, p^*; d_p)} \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \\
&\quad= \min_{\mathcal{J}\subseteq\mathcal{J}^*} \min_{d_x\in\mathbb{L}_{\mathcal{J}}(x^*, p^*; d_p)} \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \\
&\quad= \min_{\mathcal{J}\subseteq\mathcal{J}^*} \max_{(\lambda,\mu,u,v)\in\mathcal{B}_{\mathcal{J}}(x^*, p^*)} \nabla_p \mathscr{L}^1(x^*, p^*; \lambda, \mu, u, v)^T d_p. \qquad (3.18)
\end{aligned}$$

Therefore, we have from (3.16) and (3.18) that

$$\begin{aligned}
\mathcal{D}\mathcal{V}(p^*; d_p) &= \min_{x^*\in\mathcal{O}(p^*)} \min_{d_x\in\mathbb{L}(x^*, p^*; d_p)} \nabla f(x^*, p^*)^T \begin{pmatrix} d_x \\ d_p \end{pmatrix} \\
&= \min_{x^*\in\mathcal{O}(p^*)} \min_{\mathcal{J}\subseteq\mathcal{J}^*} \max_{(\lambda,\mu,u,v)\in\mathcal{B}_{\mathcal{J}}(x^*, p^*)} \nabla_p \mathscr{L}^1(x^*, p^*; \lambda, \mu, u, v)^T d_p.
\end{aligned}$$

The proof is complete. ∎

The requirement $d_p \in \cap_{x^* \in \mathcal{O}(p^*)} \mathrm{dom}\, D_+\mathcal{X}_T(x^*, p^*; \cdot)$ in Theorem 3.9 restricts the range of the differentiable directions. We next investigate how to expand the range. To this end, we need the following lemma. The result extends the metric regularity result from the case of additive perturbation, i.e., when $F(x,p) := F(x) - p$ (see, e.g., [27, Example 9.44]), to the case of nonadditive perturbation.

LEMMA 3.10 (see [9, Lemma 3.1]). *If the MPEC-NNAMCQ holds at $x^* \in \mathcal{X}(p^*)$, i.e.,*

$$\begin{cases} \nabla_x F(x^*, p^*) y = 0, \\ y \in \mathcal{N}_\Lambda(F(x^*, p^*)) \end{cases} \Longrightarrow y = 0,$$

then there exist $\delta > 0$ and $\kappa > 0$ such that

$$\mathrm{dist}(x, \mathcal{X}(p)) \leq \kappa\, \mathrm{dist}(F(x,p), \Lambda) \quad \forall x \in \mathcal{B}_\delta(x^*), \forall p \in \mathcal{B}_\delta(p^*).$$

THEOREM 3.11. *Assume that the restricted inf-compactness holds around p^*. Suppose further that the MPEC-RCR regularity and the MPEC-NNAMCQ hold at each $x^* \in \mathcal{O}(p^*)$. Then the value function $\mathcal{V}$ is directionally differentiable at $p = p^*$ in every direction $d_p \in \Re^{n_2}$ and the formula* (3.10) *holds.*

Proof. From the proof of Theorem 3.9, we know that the requirement

$$d_p \in \cap_{x^* \in \mathcal{O}(p^*)} \mathrm{dom}\, D_+\mathcal{X}_T(x^*, p^*; \cdot)$$

is to ensure that $D_+\mathcal{X}_T(x^*, p^*; d_p) \neq \emptyset$ for each $x^* \in \mathcal{O}(p^*)$ and then to make use of the fact that $D_+\mathcal{X}(x^*, p^*; d_p) = \mathbb{L}(x^*, p^*; d_p)$. Since the MPEC-NNAMCQ holds, we get from Lemmas 3.3 and 3.10 that

$$D_+\mathcal{X}(x^*, p^*; d_p) = \mathbb{L}(x^*, p^*; d_p) \quad \forall d_p \in \Re^{n_2}.$$

Moreover, it follows from [27, Exercise 6.39] that

$$\nabla_x F(x^*, p^*)^T \Re^{n_1} + \mathcal{T}_\Lambda(F(x^*, p^*)) = \Re^{m_1+m_2+2m}.$$

This implies that for any given d_p, there exists d_x such that

$$\nabla_x F(x^*, p^*)^T d_x + \nabla_p F(x^*, p^*)^T d_p \in \mathcal{T}_\Lambda(F(x^*, p^*)).$$

That is to say, $\mathbb{L}(x^*, p^*; d_p) \neq \emptyset$. Thus, the desired result is obtained easily from the proof of Theorem 3.9. ∎

Formula (3.10) has an explicit combinatorial construction corresponding to complementarity constraints. Obviously, $\mathcal{M}_S^1(x^*, p^*) \subseteq \mathcal{B}_{\mathcal{J}}(x^*, p^*)$ for each $\mathcal{J} \subseteq \mathcal{J}^*$. Thus, we have the following lower estimate for the directional derivative:

$$\min_{x^* \in \mathcal{O}(p^*)} \max_{y \in \mathcal{M}_S^1(x^*, p^*)} \nabla_p \mathscr{L}^1(x^*, p^*; y)^T d_p \leq \mathcal{D}\mathbb{V}(p^*; d_p).$$

We next give a result which relieves the combinatorial aspects under the MPEC-LICQ.

COROLLARY 3.12. *Assume that the restricted inf-compactness holds around p^*. Let the MPEC-LICQ hold at each $x^* \in \mathcal{O}(p^*)$. Then the value function is directionally differentiable at $p = p^*$ in every direction $d_p \in \Re^{n_2}$ and*

$$\mathcal{D}\mathcal{V}(p^*; d_p) = \min_{x^* \in \mathcal{O}(p^*)} \nabla_p \mathscr{L}^1(x^*, p^*; \lambda, \mu, u, v)^T d_p,$$

where (λ, μ, u, v) is the unique S-multiplier vector at $x^* \in \mathcal{X}(p^*)$.

Proof. It is not difficult to see that the MPEC-LICQ implies the MPEC-RCR regularity and MPEC-NNAMCQ. Thus, (3.10) holds. Moreover, since the MPEC-LICQ holds at $x^* \in \mathcal{O}(p^*)$, it is easy to get that $\mathcal{B}_{\mathcal{J}}(x^*, p^*) = \{(\lambda, \mu, u, v)\}$ for each $\mathcal{J} \subseteq \mathcal{J}^*$ (see [28, Theorem 4]), where (λ, μ, u, v) is actually the unique S-multiplier. The proof is complete. ∎

Corollary 3.12 is similar to [11, Theorem 2], which requires the MPEC-LICQ and the following inf-compactness assumption.

DEFINITION 3.13. *We say that inf-compactness holds for (MPEC$_p$) around p^* if there exist two positive numbers $\{\alpha, \delta\}$ and a bounded set S such that $\alpha > \mathcal{V}(p^*)$ and*

$$\{x \in \mathcal{X}(p) \,|\, f(x,p) \le \alpha,\ p \in \mathcal{B}_\delta(p^*)\} \subseteq S.$$

It is not difficult to verify that the *inf-compactness* in Definition 3.13 is stronger than the *restricted inf-compactness* in Definition 3.8. In fact, we can show that the inf-compactness and the MPEC-LICQ imply the nonemptiness and uniform compactness of $\mathcal{O}(p)$ around p^*. In the following, we compare the inf-compactness with the nonemptiness and uniform compactness of $\mathcal{O}(p)$.

THEOREM 3.14. *Let $x^* \in \mathcal{O}(p^*)$. The MPEC-LICQ at x^* and the inf-compactness around p^* imply the nonemptiness and uniform compactness of $\mathcal{O}(p)$ around p^*. Conversely, if there exists a sequence $p^k \to p^*$ such that $\mathbb{V}(p^k) \downarrow \mathbb{V}(p^*)$, then the nonemptiness and uniform compactness of $\mathcal{O}(p)$ around p^* imply the inf-compactness around p^*.*

Proof. First, we observe that, by the continuity of the objective function f, if $\emptyset \ne \{x \in \mathcal{X}(p) \,|\, f(x,p) \le \beta\} \subseteq \Delta$, then the solution set

$$\emptyset \ne \mathcal{O}(p) \subseteq \{x \in \mathcal{X}(p) \,|\, f(x,p) \le \beta\} \subseteq \Delta.$$

Thus, to show the first part, by the inf-compactness around p^*, it suffices to show that there exists $\delta_0 \in (0, \delta]$ such that the set $\{x \in \mathcal{X}(p) \,|\, f(x,p) \le \alpha\} \ne \emptyset$ for each $p \in \mathcal{B}_{\delta_0}(p^*)$, where α and δ are defined as in Definition 3.13. We now show that, under the MPEC-LICQ, the set $\{x \in \mathcal{X}(p) \,|\, f(x,p) \le \alpha\} \ne \emptyset$ for each $p \in \mathcal{B}_{\delta_0}(p^*)$. Consider the system

$$\left\{ \begin{array}{l} g(x,p) \le 0,\ h(x,p) = 0, \\ G_{\mathcal{I}^* \cup \mathcal{J}^*}(x,p) = 0,\ H_{\mathcal{K}^* \cup \mathcal{J}^*}(x,p) = 0. \end{array} \right. \tag{3.19}$$

Since the MPEC-LICQ holds at x^* for $p = p^*$, by the implicit function theorem, there exist neighborhoods V_1 and V_2 of p^* and x^*, respectively, and a smooth function $x(\cdot) : V_1 \to V_2$ such that $x(p^*) = x^*$ and $x(p)$ satisfies the above system (3.19). Then, it is easy to see that $x(p) \in \mathcal{X}(p)$ and $f(x(p), p) \to f(x^*, p^*) < \alpha$ as $p \to p^*$. Thus, there exists $\delta_0 \in (0, \delta]$ such that, for each fixed $p \in \mathcal{B}_{\delta_0}(p^*)$, the set $\{x \in \mathcal{X}(p) \,|\, f(x,p) \le \alpha\}$ is nonempty. Consequently, the nonemptiness and boundedness assumption holds.

We next show the converse part. Assume to the contrary that the inf-compactness does not hold. That is, for each $\alpha^k \downarrow \mathbb{V}(p^*)$, there exist $p^k \to p^*$ and $x^k \in \mathcal{X}(p^k)$ such that $f(x^k, p^k) \le \alpha^k$ and $\|x^k\| \to \infty$ as $k \to \infty$. By the assumption, we let $\alpha^k = \mathbb{V}(p^k)$ and, by the nonemptiness and boundedness assumption, we let $x^k \in \mathcal{X}(p^k)$ such that $f(x^k, p^k) = \alpha^k$ and $\{x^k\}$ is bounded, which gives a contradiction. The proof is complete. ∎

We next give an example to show that the assumption of the second part of Theorem 3.14 is necessary even for nonlinear programs and then show that Corollary 3.12 may be applied to more cases than [11, Theorem 2].

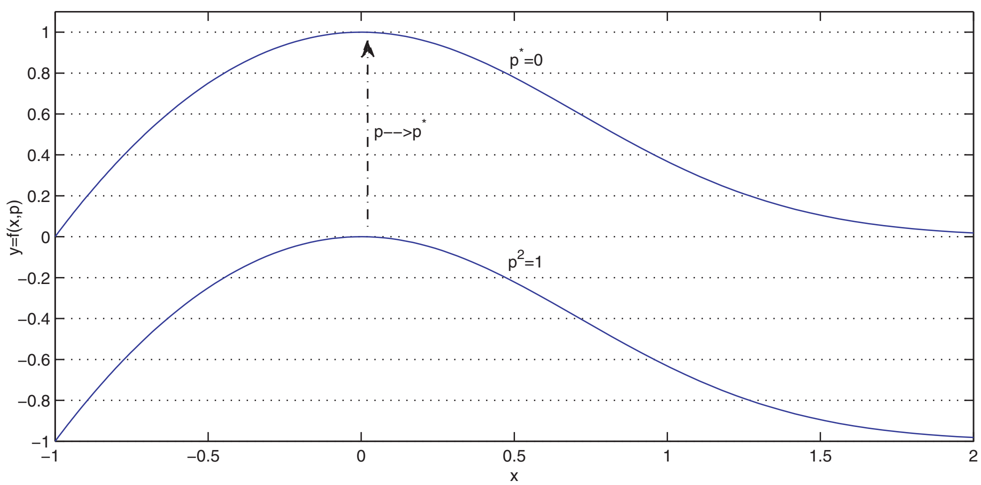

FIG. 1. *Example* 3.15.

Example 3.15. Consider the problem

$$\begin{array}{ll} \min\limits_{x} & f(x,p) := f(x) - p^2 \\ \text{s.t.} & x+1 \geq 0, \end{array}$$

where

$$f(x) := \begin{cases} -x^2+1 & \text{if } x \leq 0 \\ e^{-x^2} & \text{if } x \geq 0 \end{cases}$$

(see Figure 1). Clearly, f is C^1 in x. Moreover for each p, the LICQ holds at the unique optimal solution $x^* = -1$. Thus, the nonemptiness and uniform compactness condition holds. However, the optimal value function $\mathbb{V}(p) = -p^2$ is increasing as $p \to 0$ and hence the assumption of the second part of Theorem 3.14 fails. Moreover, it is easy to see that, for any $\alpha > \mathbb{V}(0)$, the α-level set of $f(x,p)$ is unbounded for any p and hence the inf-compactness around $p^* = 0$ does not hold. Thus, [11, Theorem 2] fails to be applied to this situation but by Corollary 3.12, we can have the directional differentiability of the value function.

Theorem 3.14 and Example 3.15 show that the nonemptiness and uniform compactness of the optimal solution mapping and the MPEC-LICQ are strictly weaker than the inf-compactness and the MPEC-LICQ. Moreover, it is easy to see that the restricted inf-compactness is strictly weaker than the nonemptiness and uniform compactness of the optimal solution mapping. Therefore, Corollary 3.12 improves [11, Theorem 2].

In the rest of this section, we study the differentiability of the localized optimal value function. The following definition and lemmas will be useful.

DEFINITION 3.16 (see [8]). *We say that the* S-multiplier refined second-order sufficient condition *(S-RSOSC) holds at* $x^* \in \mathcal{X}(p^*)$ *if, for every* $d \in \mathcal{C}(x^*,p^*)\backslash\{0\}$, *there exist* $r \geq 0$ *and* $y^* \in \mathcal{M}_S^r(x^*,p^*)$ *such that*

$$d^T \nabla_x^2 \mathscr{L}^r(x^*,p^*;y^*)d > 0,$$

where $\mathcal{C}(x^*,p^*) := \mathcal{L}(x^*,p^*) \cap \{d \mid \nabla_x f(x^*,p^*)^T d \leq 0\}$.

LEMMA 3.17 (see [8]). *Let $x^* \in \mathcal{X}(p^*)$. If the S-RSOSC holds at x^*, then x^* satisfies the second-order growth condition, i.e., there exist $\delta > 0$ and $c > 0$ such that*

$$f(x, p^*) \geq f(x^*, p^*) + c\|x - x^*\|^2 \qquad \forall x \in \mathcal{X}(p^*) \cap \mathcal{B}_\delta(x^*).$$

LEMMA 3.18 (see [9]). *Let $x^* \in \mathcal{X}(p^*)$ be an M-stationary point of (MPEC$_{p^*}$). Suppose that the MPEC relaxed constant positive linear dependence (MPEC-RCPLD) and the second-order growth condition hold at x^*. Then there exists a neighborhood V of x^* containing no other M-stationary points of (MPEC$_{p^*}$).*

The MPEC-RCPLD, which was introduced in [8] and was shown to be a constraint qualification for M-stationarity in [7], is weaker than the MPEC relaxed constant rank constraint qualificiation (MPEC-RCRCQ) and MPEC-NNAMCQ.

From Lemmas 3.17–3.18, we have the following result immediately.

THEOREM 3.19. *Let $x^* \in \mathcal{X}(p^*)$. Suppose that the MPEC-NNAMCQ and S-RSOSC hold at x^*. Then there exists $\delta_0 > 0$ such that the localized MPEC*

$$\begin{aligned} \min_x \quad & f(x, p) \\ \text{s.t.} \quad & x \in \mathcal{X}(p), \ \|x - x^*\|^2 \leq \delta_0, \end{aligned} \tag{3.20}$$

has a uniquely and globally optimal solution x^* for $p = p^*$. Furthermore, there exists $\bar{\delta} > 0$ such that the set of globally optimal solutions of (3.20) *is nonempty and is contained in $\bar{\mathcal{B}}_{\sqrt{\delta_0}}(x^*)$ for each $p \in \mathcal{B}_{\bar{\delta}}(p^*)$.*

Proof. Since the S-RSOSC holds at x^*, by Lemma 3.17, there exists $\delta_1 > 0$ such that x^* is a global minimizer of (MPEC$_{p^*}$) over $\bar{\mathcal{B}}_{\delta_1}(x^*)$. Since the MPEC-NNAMCQ persists under small perturbations, there exists $\delta_2 \in (0, \delta_1]$ such that the MPEC-NNAMCQ holds over $\bar{\mathcal{B}}_{\delta_2}(x^*) \cap \mathcal{X}(p)$. By Lemma 3.18, we pick $\delta_0 \in (0, \delta_2/2]$ such that there is no other M-stationary points over $\bar{\mathcal{B}}_{\delta_0}(x^*)$. If problem (3.20) has another minimizer $x_0 \neq x^*$ when $p = p^*$, then $f(x_0) = f(x^*)$. Thus, x_0 is a global minimizer of (MPEC$_p$) over $\bar{\mathcal{B}}_{\delta_1}(x^*)$ and by $\delta_0 \leq \frac{\delta_1}{2}$, it is easy to see that x_0 is a local minimizer of (MPEC$_{p^*}$), which implies that it is an M-stationary point by the fact that the MPEC-NNAMCQ implies that any local minimizer is an M-stationary point. Thus, problem (3.20) has a uniquely and globally optimal solution x^* for $p = p^*$. Since the MPEC-NNAMCQ holds at x^*, it follows from Lemma 3.10 that there exist $\kappa > 0$ and $\delta > 0$ such that, for each $p \in \mathcal{B}_\delta(p^*)$,

$$\begin{aligned} \operatorname{dist}(x^*, \mathcal{X}(p)) &\leq \kappa \operatorname{dist}(F(x^*, p), \Lambda) \\ &\leq \kappa \|F(x^*, p) - F(x^*, p^*)\|. \end{aligned}$$

It follows from the continuity of F in p and the last inequality that there exists $\bar{\delta} > 0$ such that, for each $p \in \mathcal{B}_{\bar{\delta}}(p^*)$, $\operatorname{dist}(x^*, \mathcal{X}(p) \leq \delta_0$. Thus, the feasible region of problem (3.20) is nonempty and hence the desired result is obtained. The proof is complete. ∎

We denote by $\mathbb{W}$ and $\mathbb{O}$ the value function and the optimal solution mapping of problem (3.20), respectively. Note that x^* is the uniquely and globally optimal solution of (3.20) for $p = p^*$ and the constraint $\|x - x^*\|^2 \leq \delta_0$ is not active at x^*. By Corollary 3.12 and Theorem 3.19, we have the following result immediately. Since the MPEC strong second order sufficient condition (MPEC-SSOSC) in the sense of [11] is much stronger than the S-SROSC as the critical cone in [11] is much bigger, the following result improves [11, Theorem 1].

COROLLARY 3.20. *Suppose that the MPEC-LICQ and S-RSOSC hold at $x^* \in \mathcal{X}(p^*)$. Then $\mathbb{W}$ is differentiable at $p = p^*$ and*

$$\nabla \mathbb{W}(p^*) = \nabla_p \mathscr{L}^1(x^*, p^*; \lambda, \mu, u, v),$$

where (λ, μ, u, v) is the unique S-multiplier vector at $x^* \in \mathcal{X}(p^*)$.

Proof. It follows from Theorem 3.19 that $\mathbb{O}(p^*) = \{x^*\}$ and the restricted inf-compactness holds. Since the constraint $\|x - x^*\|^2 \leq \delta_0$ is not active at x^*, the MPEC-LICQ holds at x^* for (3.20). Thus, we have the desired result from Corollary 3.12 immediately. ☐

From Corollary 3.20, we have the following result, which means that the S-multipliers are actually the shadow prices as in NLP.

COROLLARY 3.21. *Let*

$$\mathcal{X}(p) := \{x \mid g(x) + p^g \leq 0,\ h(x) + p^h = 0,\ 0 \leq G(x) + p^G \perp H(x) + p^H \geq 0\}$$

and $f(x, p) := f(x)$. Assume that the MPEC-LICQ and S-RSOSC hold at $x^* \in \mathcal{X}(p^*)$. Then we have

$$\nabla \mathbb{W}(p^*) = (\lambda, \mu, -u, -v),$$

where (λ, μ, u, v) is the unique S-multiplier vector at $x^* \in \mathcal{X}(p^*)$.

4. Subdifferential of the value function.

For (MPEC$_p$), Lucet and Ye [15, 16] gave upper estimates for the singular subdifferential and limiting subdifferential of the value function by C-, M-, and S-multipliers under the conditions that the growth condition and some normality conditions hold. In this section, we obtain some sharper upper estimates for the singular subdifferential and the limiting subdifferential of the value function for (MPEC$_p$) based on the enhanced Fritz John condition for MPECs under the weaker conditions that the restricted inf-compactness and some quasi-normality conditions hold. For the sake of simplicity, we denote

$$\begin{aligned}\partial_x \mathscr{L}^r(x, p; \lambda, \mu, u, v) := r\partial_x f(x, p) + \sum_{i=1}^{m_1} \lambda_i \partial_x g_i(x, p) + \sum_{j=1}^{m_2} \partial_x(\mu_j h_j)(x, p) \\ + \sum_{l=1}^{m} \partial_x(u_l G_l)(x, p) + \sum_{l=1}^{m} \partial_x(v_l H_l)(x, p).\end{aligned}$$

Note that the limiting subdifferential of $\mathscr{L}^r$ at $(x, p, \lambda, \mu, u, v)$ with respect to x is not equal to the right-hand side of the above equation and, for simplicity, we use all plus signs in the formula above in contrast with the standard MPEC Lagrangian function. Moreover, in this section, we assume that all the involved functions $\{f, g, h, G, H\}$ are only Lipschitzian around the point of interest.

4.1. Subdifferential via enhanced M-multipliers.

Let us first give the enhanced Fritz John-type M-stationary condition for (MPEC$_{p^*}$). In fact, Kanzow and Schwartz [14, Theorem 3.1] have presented the smooth enhanced Fritz John-type M-stationary condition. In the following, we show that, for the nonsmooth case, any local minimizer for MPEC is also an enhanced Fritz John-type M-stationary point.

An enhanced Fritz John optimality condition is given for a very general mathematical program with geometric constraints in Banach spaces in [10, Theorem 3.1 and

Corollaries 3.1–3.2]. We next specialize this result to MPEC. By means of (3.3)–(3.4), (MPEC_p) can be rewritten as the more compact form

$$\min_{x\in\mathcal{C}} \quad f(x,p) \qquad \text{s.t.} \quad F(x,p)\in\Lambda. \tag{4.1}$$

Since problem (4.1) is the problem of the form considered in [10], we can specialize [10, Corollary 3.3] to problem (4.1).

THEOREM 4.1. *Let $x^*\in\mathcal{X}(p^*)$ be a local minimizer of* (MPEC_{p^*}). *Then there exists $0\neq(r,\lambda^*,\mu^*,u^*,v^*)$ with $r\geq 0$ such that*

(1) $0\in\partial_x\mathscr{L}^r(x^*,p^*;\lambda^*,\mu^*,u^*,v^*)+\mathcal{N}_{\mathcal{C}}(x^*)$, $\lambda^*\geq 0$, $\lambda_i^*=0$ $(i\notin I_g^*)$, $u_i^*=0$ $(i\in\mathcal{K}^*)$, $v_i^*=0$ $(i\in\mathcal{I}^*)$, $u_i^*v_i^*=0$ or $u_i^*<0$, $v_i^*<0$ $(i\in\mathcal{J}^*)$;

(2) *there exists a sequence $\{x^k\}\subset\mathcal{C}$ converging to x^* such that, for each k,*

$$\lambda_i^*>0\Longrightarrow\lambda_i^*g_i(x^k,p^*)>0,\quad \mu_i^*>0\Longrightarrow\mu_ih_i(x^k,p^*)>0,$$
$$u_i^*\neq 0\Longrightarrow u_i^*G_i(x^k,p^*)>0,\quad v_i^*\neq 0,\Longrightarrow v_i^*H(x^k,p^*)>0,$$

and $\{h,g,G,H\}$ are all differentiable with respect to x at (x^k,p^*).

Proof. Since x^* is a local minimizer of (4.1) for $p=p^*$, by [10, Corollary 3.3], there exist a scalar $r\geq 0$ and a vector η^*, not all zero, such that the following conditions hold, where $\{e_i\mid i=1,\dots,m_1+m_2+2m\}$ is the orthogonal basis of $\Re^{m_1+m_2+2m}$:

(i) $0\in r\partial_xf(x^*,p^*)+\sum_{i=1}^{m_1+m_2+2m}\partial_x\langle\eta^*,e_i\rangle\langle F,e_i\rangle(x^*,p^*)+\mathcal{N}_{\mathcal{C}}(x^*)$;

(ii) $\eta^*\in\mathcal{N}_\Lambda(F(x^*,p^*))$;

(iii) there exists a sequence $\{(x^k,y^k,\eta^k)\}\subset\mathcal{C}\times\Lambda\times\Re^{m_1+m_2+2m}$ converging to $(x^*,F(x^*),\eta^*)$ such that, for all k,

$$f(x^k,p^*)<f(x^*,p^*),$$
$$\eta^k\in\mathcal{N}_\Lambda(y^k), \tag{4.2}$$
$$\langle\eta^*,e_i\rangle\neq 0\Longrightarrow\langle\eta^*,e_i\rangle\langle F(x^k,p^*)-y^k,e_i\rangle>0. \tag{4.3}$$

Let $\eta:=(\lambda,\mu,u,v)$ and $y:=(y^1,y^2,y^3,y^4)$ with appropriate dimensional components corresponding to (f,h,G,H). By the explicit expression of limiting normal cone $\mathcal{N}_\Lambda$ (see, e.g., [9, Proposition 5.1]), we have (1) immediately. We next show (2). It follows from (iii) that $\{x^k\}\subseteq\mathcal{C}$, $\{y^{1,k}\}\subseteq\Re_-^{m_1}$, $\{y^{2,k}\}=\{0\}^{m_2}$, $\{(y^{3,k},y^{4,k})\}\subseteq C^m$, and

$$\lambda_i^*>0\Longrightarrow\lambda_i^*(g_i(x^k,p^*)-y_i^{1,k})>0,\ \mu_i^*>0\Longrightarrow\mu_i^*(h_i(x^k,p^*)-y_i^{2,k})>0, \tag{4.4}$$
$$u_i^*\neq 0\Longrightarrow u_i^*(G_i(x^k,p^*)-y_i^{3,k})>0,\ v_i^*\neq 0,\Longrightarrow v_i^*(H(x^k,p^*)-y_i^{4,k})>0. \tag{4.5}$$

Next, we show that, for each sufficiently large k, $\{y_i^{1,k},y_i^{2,k},y_i^{3,k},y_i^{4,k}\}$ in (4.4)–(4.5) is equal to 0. Assume to the contrary that there exists a subsequence such that it does not hold. We first notice that $y^{2,k}=0$. Thus, we consider the following three cases.

- If $y_i^{1,k}<0$ for a subsequence $K_1\subseteq\{1,2,\dots\}$, then $\mathcal{N}_{\Re_-}(y_i^{1,k})=\{0\}$ $\forall k\in K_1$. Thus it follows from (4.2) that $\lambda_i^k\to\lambda_i^*=0$ as $K_1\ni k\to\infty$, which contradicts $\lambda_i^*>0$.
- If $y_i^{3,k}>0$ for a subsequence $K_2\subseteq\{1,2,\dots\}$, then $y_i^{4,k}=0$ $\forall k\in K_2$. Thus, it follows from the explicit expression of $\mathcal{N}_C$ that $u_i^k\to u_i^*=0$ as $K_2\ni k\to\infty$, which contradicts $u_i^*\neq 0$.
- Similarly as above we can show that it is impossible to have $y_i^{4,k}<0, k\in K_3$ for some subsequence $K_3\subseteq\{1,2,\dots\}$.

So far we have shown (2) except for the differentiability of $\{h, g, G, H\}$ with respect to x at (x^k, p^*). By Rademacher's theorem, if a function ψ is Lipschitzian around x^*, then ψ is differentiable almost everywhere around x^*. Based on this fact, if $x^k \to x^*$ and $\psi(x^k) > 0$, then one can always find a sequence $\{\bar{x}^k\}$ with $\psi(\bar{x}^k) > 0$ such that for all k, ψ is differentiable at $\bar{x}^k$ and $\|\bar{x}^k - x^k\| \le \frac{1}{k}$. Hence we have shown that the sequence $\{\bar{x}^k\}$ satisfies the condition (2). The proof of the theorem is complete by resetting x^k with $\bar{x}^k$ for each k. $\square$

DEFINITION 4.2. *Given $r \ge 0$ and $x^* \in \mathcal{X}(p^*)$, we let $\mathscr{M}_M^r(x^*, p^*)$ denote the set of vectors $(\lambda, \mu, u, v, \zeta)$ such that*

(i) $0 \in \partial_{(x,p)} \mathscr{L}^r(x^*, p^*; \lambda, \mu, u, v) - (0, \zeta) + \mathcal{N}_{\mathcal{C}}(x^*) \times \{0\}$;

(ii) $\lambda \ge 0,\ \lambda_i = 0\ (i \notin I_g^*),\ u_i = 0\ (i \in \mathcal{K}^*),\ v_i = 0\ (i \in \mathcal{I}^*),\ u_i v_i = 0$ or $u_i < 0,\ v_i < 0\ (i \in \mathcal{J}^*)$;

(iii) *there exists a sequence $\{(x^k, p^k)\} \subseteq \mathcal{C} \times \Re^{n_2}$ converging to (x^*, p^*) such that, for each k,*

$$\lambda_i > 0 \Longrightarrow \lambda_i g_i(x^k, p^k) > 0, \quad \mu_j \neq 0 \Longrightarrow \mu_j h_j(x^k, p^k) > 0,$$
$$u_l \neq 0 \Longrightarrow u_l G_l(x^k, p^k) > 0, \quad v_l \neq 0 \Longrightarrow v_l H_l(x^k, p^k) > 0,$$

and $\{h, g, G, H\}$ are all differentiable at (x^k, p^k).

In order to study the subdifferential of the value function of (MPEC_p), the following constraint qualification for ghp $\mathcal{X}$ will be useful.

DEFINITION 4.3. *Let $x^* \in \mathcal{X}(p^*)$. We say that the MPEC M-quasi-normality holds at $x^* \in \mathcal{X}(p^*)$ if $(\lambda, \mu, u, v, 0) \in \mathscr{M}_M^0(x^*, p^*) \Longrightarrow (\lambda, \mu, u, v) = \{0\}$.*

LEMMA 4.4. *Let $x^* \in \mathcal{O}(p^*)$. Assume that (p^*, x^*) is MPEC M-quasi-normal for the constraint region* gph $\mathcal{X}$. *Then the following upper estimate holds:*

$$\hat{\partial}\mathcal{V}(p^*) \subseteq \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_M^1(x^*, p^*)\}. \tag{4.6}$$

Proof. Let $\zeta \in \hat{\partial}\mathcal{V}(p^*)$. Then, by the definition of the Fréchet subdifferential, for an arbitrary $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\mathcal{V}(p) - \mathcal{V}(p^*) \ge \zeta^T(p - p^*) - \epsilon\|p - p^*\| \quad \forall p \in \mathcal{B}_{\delta_\epsilon}(p^*).$$

By the definition of value function, we have $f(x, p) \ge \mathcal{V}(p)$ for every $x \in \mathcal{X}(p)$ and hence

$$f(x, p) - \zeta^T(p - p^*) + \epsilon\|p - p^*\| \ge f(x^*, p^*) \quad \forall x \in \mathcal{X}(p), \forall p \in \mathcal{B}_{\delta_\epsilon}(p^*).$$

Thus, (x^*, p^*) is a locally optimal solution of the optimization problem

$$\begin{array}{ll} \min & f(x, p) - \zeta^T(p - p^*) + \epsilon\|p - p^*\| \\ \text{s.t.} & g(x, p) \le 0,\ h(x, p) = 0, \\ & 0 \le G(x, p) \ \perp\ H(x, p) \ge 0, \\ & (x, p) \in \mathcal{C} \times \Re^{n_2}. \end{array}$$

By Theorem 4.1 and the MPEC M-quasi-normality assumption, there exists a vector (λ, μ, u, v) such that the following conditions hold:

(i) $0 \in \partial_{(x,p)} \mathscr{L}^1(x^*, p^*, \lambda, \mu, u, v) - (0, \zeta) + \mathcal{N}_{\mathcal{C}}(x^*) \times \{0\} + \epsilon \begin{pmatrix} 0 \\ \bar{\mathcal{B}} \end{pmatrix}$, $\lambda \ge 0$, $\lambda_i = 0\ (i \notin I_g^*)$, $u_i = 0\ (i \in \mathcal{K}^*)$, $v_i = 0\ (i \in \mathcal{I}^*)$, $u_i v_i = 0$ or $u_i < 0$, $v_i < 0\ (i \in \mathcal{J}^*)$;

(ii) there exists a sequence $\{(x^k, p^k)\} \subseteq \mathcal{C} \times \Re^{n_2}$ converging to (x^*, p^*) such that, for each k,

$$\lambda_i > 0 \Longrightarrow \lambda_i g_i(x^k, p^k) > 0, \quad \mu_j \neq 0 \Longrightarrow \mu_j h_j(x^k, p^k) > 0,$$
$$u_l \neq 0 \Longrightarrow u_l G_l(x^k, p^k) > 0, \quad v_l \neq 0 \Longrightarrow v_l H_l(x^k, p^k) > 0,$$

and $\{h, g, G, H\}$ are all differentiable at (x^k, p^k).

The desired upper estimate follows since ϵ is arbitrary. ∎

We now give a tighter estimate for the limiting subdifferential of the value function in terms of the enhanced M-multipliers than the one given in [15, 16]. To this end, we first give several lemmas.

The following lemma is similar to [32, Lemma 1] and [10, Proposition 6.2].

LEMMA 4.5. *If a vector (p^*, x^*) is MPEC M-quasi-normal for the constraint region* $\operatorname{gph}\mathcal{X}$*, then there exists a neighborhood V of (p^*, x^*) such that all vectors $(p, x) \in \operatorname{gph}\mathcal{X} \cap V$ are MPEC M-quasi-normal.*

The following lemma can be obtained from the proof of [15, Lemma 3.4].

LEMMA 4.6. *Assume that $\varphi : \Re^n \to \Re \cup \{\infty\}$ is Lipschitzian around x^*. If $u^k \to u^*$, $v^k \to v^*$, and $x^k \to x^*$ with $v^k \to \partial(u^k\varphi)(x^k)$, then $v^* \in \partial(u^*\varphi)(x^*)$.*

THEOREM 4.7. *Assume that the restricted inf-compactness holds for (MPEC$_p$) around p^*. Then the value function $\mathcal{V}(p)$ is lower semicontinuous at p^*. Suppose further that, for each $x^* \in \mathcal{O}(p^*)$, (p^*, x^*) is MPEC M-quasi-normal for the constraint region* $\operatorname{gph}\mathcal{X}$*. Then*

$$\partial\mathcal{V}(p^*) \subseteq \bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_M^1(x^*, p^*)\},$$
$$\partial^\infty\mathcal{V}(p^*) \subseteq \bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_M^0(x^*, p^*)\}.$$

Proof. The lower semicontinuity follows from the restricted inf-compactness immediately [3, Page 246] (see also the proof of Theorem 3.9). We complete the proof by considering the following two cases.

(a) Let $\zeta \in \partial\mathcal{V}(p^*)$. By the definition, there exist sequences $p^l \to_{\mathcal{V}} p^*$ and $\zeta^l \to \zeta$ with $\zeta^l \in \hat{\partial}\mathcal{V}(p^l)$. Since the restricted inf-compactness holds, $\mathcal{V}(p^*)$ is finite. Since $\mathcal{V}(p^l) \to \mathcal{V}(p^*)$, we have $\mathcal{V}(p^l) < \mathcal{V}(p^*) + \epsilon_0$ for each sufficiently large l. By the restricted inf-compactness again, there exists $x^l \in \mathcal{O}(p^l)$ for each sufficiently large l and $\{x^l\}$ is bounded. Without loss of generality, we assume that $x^l \to x^*$. Since

$$\mathcal{V}(p^*) \leftarrow \mathcal{V}(p^l) = f(x^l, p^l) \to f(x^*, p^*), \quad k \to \infty,$$

we have $f(x^*, p^*) = \mathcal{V}(p^*)$ and hence $x^* \in \mathcal{O}(p^*)$. Since the MPEC M-quasi-normality holds at (x^*, p^*) and $(x^l, p^l) \to (x^*, p^*)$, by Lemma 4.5, the MPEC M-quasi-normality holds at (x^l, p^l) for each sufficiently large l. Thus, it follows from Lemma 4.4 that, for each sufficiently large l, there exists a vector $(\lambda^l, \mu^l, u^l, v^l)$ such that

(i) $(0, \zeta^l) \in \partial_{(x,p)}\mathscr{L}^1(x^l, p^l; \lambda^l, \mu^l, u^l, v^l) + \mathcal{N}_{\mathcal{C}}(x^l) \times \{0\}$;

(ii) $\lambda^l \geq 0$, $\lambda_i^l = 0$ $(i \notin I_g^l)$, $u_i^l = 0$ $(i \in \mathcal{K}^l)$, $v_i^l = 0$ $(i \in \mathcal{I}^l)$, $u_i^l v_i^l = 0$ or $u_i^l < 0$, $v_i^l < 0$ $(i \in \mathcal{J}^l)$;

(iii) there exists a sequence $\{(x^{l,k}, p^{l,k})\}_k$ converging to (x^l, p^l) as $k \to \infty$ such that

$$\lambda_i^l > 0 \Longrightarrow \lambda_i^l g_i(x^{l,k}, p^{l,k}) > 0, \quad \mu_j^l \neq 0 \Longrightarrow \mu_j^l h_j(x^{l,k}, p^{l,k}) > 0,$$
$$u_\imath^l \neq 0 \Longrightarrow u_\imath^l G_\imath(x^{l,k}, p^{l,k}) > 0, \quad \nu_\jmath^l \neq 0 \Longrightarrow v_\jmath^l H_\jmath(x^{l,k}, p^{l,k}) > 0,$$

and $\{h, g, G, H\}$ are all continuously differentiable at $(x^{l,k}, p^{l,k})$, where $\{I_g^l, \mathcal{I}^l, \mathcal{J}^l, \mathcal{K}^l\}$ are index sets corresponding to (x^l, p^l). Since (p^*, x^*) is MPEC M-quasi-normal, by using the reduction to absurdity, we can show that the sequence $\{(\lambda^l, \mu^l, u^l, v^l)\}$ is bounded (see, e.g, [32, Theorem 3] or [10, Proposition 6.3]). Thus, without loss of generality, we may assume that $\{(\lambda^l, \mu^l, u^l, v^l)\}$ converges to (λ, μ, u, v). Taking a limit in (i)–(ii) above and noting that (ii) is the more compact form $(\lambda^l, u^l, v^l) \in \mathcal{N}_{\mathcal{R}_-^{m_1}}(g(x^l)) \times \mathcal{N}_{C^m}(G(x^l), H(x^l))$, it follows from Lemma 4.6 and the outer semicontinuity of the limiting subdifferential and limiting normal cone that

$$\begin{gathered}(0, \zeta) \in \partial_{(x,p)} \mathscr{L}^1(x^*, p^*; \lambda, \mu, u, v) + \mathcal{N}_{\mathcal{C}}(x^*) \times \{0\},\\ \lambda \geq 0,\ \lambda_i = 0\ (i \notin I_g^*), u_i = 0\ (i \in \mathcal{K}^*), v_i = 0\ (i \in \mathcal{I}^*),\\ u_i v_i = 0 \text{ or } u_i < 0,\ v_i < 0\ (i \in \mathcal{J}^*).\end{gathered}$$

Moreover, by the diagonal rule, we can find a sequence $\{(x^{l,k_l}, p^{l,k_l})\}$ converging to (x^*, p^*) as $l \to \infty$ and, for all l,

$$\begin{aligned}\lambda_i > 0 &\Longrightarrow \lambda_i g_i(x^{l,k_l}, p^{l,k_l}) > 0, \quad \mu_j \neq 0 \Longrightarrow \mu_j h_j(x^{l,k_l}, p^{l,k_l}) > 0,\\ u_\imath \neq 0 &\Longrightarrow u_\imath G_\imath(x^{l,k_l}, p^{l,k_l}) > 0, \quad v_\jmath \neq 0 \Longrightarrow v_\jmath H_\jmath(x^{l,k_l}, p^{l,k_l}) > 0,\end{aligned}$$

and $\{h, g, G, H\}$ are differentiable at (x^{l,k_l}, p^{l,k_l}). Therefore, $(\lambda, \mu, u, v, \zeta) \in \mathscr{M}_M^1(x^*, p^*)$.

(b) Let $\zeta \in \partial^\infty \mathcal{V}(p^*)$. By the definition, there exist sequences $p^l \to_{\mathcal{V}} p^*$, $\zeta^l \in \hat{\partial}\mathcal{V}(p^l)$, and $t_l \downarrow 0$ such that $t_l \zeta^l \to \zeta$. Similarly as (a), for each l sufficiently large, there exists a vector $(\lambda^l, \mu^l, u^l, v^l)$ such that (i)–(ii) hold. Multiplying (i) by t_l implies

$$(0, t_l \zeta^l) \in \partial_{(x,p)}(t_l \mathscr{L}^1)(x^l, p^l; \lambda^l, \mu^l, u^l, v^l) + \mathcal{N}_{\mathcal{C}}(x^l) \times \{0\}. \tag{4.7}$$

Since (p^*, x^*) is MPEC M-quasi-normal, by using the reduction to absurdity, we can show that the sequence $\{t_l\lambda^l, t_l\mu^l, t_l u^l, t_l v^l\}$ is bounded (see, e.g, [32, Theorem 3] or [10, Proposition 6.3]). Without loss of generality, we may assume that $\{t_l\lambda^l, t_l\mu^l, t_l u^l, t_l v^l\}$ converges to $\{\lambda, \mu, u, v\}$. Taking a limit in (4.7), we have from Lemma 4.6 and the outer semiconituity of the limiting subdifferential and limiting normal cone that

$$(0, \zeta) \in \partial_{(x,p)} \mathscr{L}^0(x^*, p^*; \lambda, \mu, u, v) + \mathcal{N}_{\mathcal{C}}(x^*) \times \{0\}.$$

The rest of the proof is similar to (a). ☐

COROLLARY 4.8. *Assume that the restricted inf-compactness holds for problem* (MPEC$_p$) *around* p^*. *Suppose that, for each* $x^* \in \mathcal{O}(p^*)$, (p^*, x^*) *is MPEC M-quasi-normal for the constraint region* gph $\mathcal{X}$. *If*

$$\bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_M^0(x^*, p^*)\} = \{0\},$$

then the value function $\mathcal{V}$ *is Lipschitzian around* p^* *with*

$$\emptyset \neq \partial\mathcal{V}(p^*) \subseteq \bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_M^1(x^*, p^*)\}.$$

In addition to the above assumptions, if

$$\bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_M^1(x^*, p^*)\} = \{\zeta\},$$

then $\mathcal{V}$ *is strictly differentiable at* p^* *and* $\nabla\mathcal{V}(p^*) = \zeta$.

Proof. Since the restricted inf-compactness holds around p^*, $\mathcal{V}$ is lower semicontinuous near p^*. Thus, it follows from Proposition 2.1 that $\mathcal{V}$ is Lipschitzian around p^*. The nonemptiness of $\partial\mathcal{V}(p^*)$ follows from Proposition 2.2 and the strict differentiability comes from Proposition 2.3. ∎

We now consider the special case where all the functions $\{f, g, h, G, H\}$ are differentiable. In this case, Definition 4.2(i) becomes

(i)^1 $0 \in \nabla_x \mathscr{L}^r(x^*, p^*; \lambda, \mu, u, v) + \mathcal{N}_{\mathcal{C}}(x^*)$; (i)^2 $\zeta = \nabla_p \mathscr{L}^r(x^*, p^*; \lambda, \mu, u, v)$.

We define the set of the singular and nonsingular enhanced M-multipliers as the set of vectors (λ, μ, u, v) satisfying (i)^1 and Definition 4.2(ii)–(iii), and denote them by $\mathcal{M}_M^r(x^*, p^*)$, $r = 0, 1$, respectively.

The following example shows that our result Theorem 4.7 is much sharper than its M-counterpart [16, Theorem 4.4].

Example 4.9. Consider the problem

$$\begin{aligned} \min_x \quad & f(x) \equiv 1 \\ \text{s.t.} \quad & g(x) := x_1 + x_2 + p \le 0, \\ & 0 \le G(x) := x_1 + p \perp H(x) := x_2 \ge 0. \end{aligned}$$

It is clear that the value function $\mathcal{V}(p) \equiv 1$ and for $p^* = 0$, the unique feasible solution $(x_1^*, x_2^*) = (0, 0)$ is the optimal solution. By solving the following singular and nonsingular M-stationarity systems for the parametric MPEC at $x^* \in \mathcal{X}(p^*)$,

$$\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\lambda \ge 0, \ u < 0, v < 0 \text{ or } uv = 0,$$

we find the sets of singular and nonsingular M-multipliers

$$\mathbb{M}^r(x^*, p^*) := \{\lambda(1, -1, -1) \mid \lambda \ge 0\}, \quad r = 0, 1.$$

Since the set of singular M-multipliers contains a nonzero vector, [16, Theorem 4.4] is not applicable and one cannot even get the Lipschitz continuity of the value function. However, for any sequence $(x^k, p^k) \to (x^*, p^*)$ and any multiplier $\lambda(1, -1, -1)$ with $\lambda > 0$, the following system of inequalities does not hold:

$$\lambda(x_1^k + x_2^k + p^k) > 0, \ -\lambda(x_1^k + p^k) > 0, \ -\lambda x_2^k > 0.$$

Thus, the sets of enhanced singular and nonsingular M-multipliers are $\mathcal{M}_M^r(x^*, p^*) = \{(0, 0, 0)\}$ for $r = 0, 1$, which are contained strictly in $\mathbb{M}^r(x^*, p^*)$ for $r = 0, 1$, respectively. Then the MPEC M-quasi-normality holds at $x^* \in \mathcal{X}(p^*)$. Since

$$\{\lambda + u \mid (\lambda, u, v) \in \mathcal{M}_M^0(x^*, p^*)\} = \{\lambda + u \mid (\lambda, u, v) \in \mathcal{M}_M^1(x^*, p^*)\} = \{0\},$$

by Corollary 4.8, we have that the value function $\mathcal{V}$ is strictly differentiable with $\nabla\mathcal{V}(p^*) = 0$.

4.2. Subdifferential via enhanced C-multipliers. In this subsection, we study the subdifferential of the value function in terms of the enhanced C-multipliers. To this end, we first give the nonsmooth enhanced Fritz John-type C-stationarity condition for MPECs.

LEMMA 4.10 (see [25, Theorems 7.5 and 7.6]). *Let*

$$g(x) := \max\{g_i(x) \mid i = 1, \dots, m\},$$

where $g_i : \Re^n \to \Re \cup \{\infty\}$, *and* $I(\bar{x}) := \{i \mid g_i(\bar{x}) = g(\bar{x})\}$. *Let* $g_i, i = 1, \ldots, m$, *be Lipschitzian around* $\bar{x}$. *Then* g *is Lipschitzian around* $\bar{x}$ *and*

$$\partial g(\bar{x}) \subseteq \bigcup \left\{ \partial \left(\sum_{i \in I(\bar{x})} \lambda_i g_i \right)(\bar{x}) \mid \sum_{i \in I(\bar{x})} \lambda_i = 1, \lambda_i \geq 0, i \in I(\bar{x}) \right\}.$$

Let $f(x) := \min\{f_i(x) \mid i = 1, \ldots, m\}$, *where* $f_i : \Re^n \to \Re \cup \{\infty\}$, *and* $J(\bar{x}) := \{i \mid f_i(\bar{x}) = f(\bar{x})\}$. *Assume that* f_i *is lower semicontinuous near* $\bar{x}$ *for* $i \in J(\bar{x})$ *and lower semicontinuous at* $\bar{x}$ *for* $i \notin J(\bar{x})$. *Then* f *is lower semicontinuous near* $\bar{x}$ *and*

$$\partial f(\bar{x}) \subseteq \bigcup \{\partial f_i(\bar{x}) \mid i \in J(\bar{x})\}.$$

THEOREM 4.11. *If* x^* *is a local minimizer of* (MPEC$_{p^*}$), *then there exist nonzero vectors* (r, λ, μ, u, v) *with* $r \geq 0$ *such that*

(i) $0 \in \partial_x \mathscr{L}^r(x^*, p^*; \lambda, \mu, u, v) + \mathcal{N}_{\mathcal{C}}(x^*)$;

(ii) $\lambda \geq 0$, $\lambda_i = 0$ $(i \notin I_g^*)$, $u_i = 0$ $(i \in \mathcal{K}^*)$, $v_i = 0$ $(i \in \mathcal{I}^*)$, $u_i v_i \geq 0$ $(i \in \mathcal{J}^*)$;

(iii) *there exists a sequence* $\{x^k\} \subseteq \mathcal{C}$ *converging to* x^* *such that, for each* k,

$$\begin{aligned}
\lambda_i > 0 &\Longrightarrow \lambda_i g_i(x^k, p^*) > 0,\\
\mu_j \neq 0 &\Longrightarrow \mu_j h_j(x^k, p^*) > 0,\\
u_l \neq 0 &\Longrightarrow u_l \min(G_l(x^k, p^*), H_l(x^k, p^*)) > 0,\\
v_l \neq 0 &\Longrightarrow v_l \min(G_l(x^k, p^*), H_l(x^k, p^*)) > 0,
\end{aligned}$$

and $\{h(\cdot, p^*), g(\cdot, p^*), \min(G_l, H_l)(\cdot, p^*)\}$ *are all differentiable at* x^k.

Proof. Since the feasible region $\mathcal{X}(p)$ of (MPEC$_p$) can be written as

$$\mathcal{X}(p) = \{x \in \mathcal{C} \mid g(x, p) \leq 0, h(x, p) = 0, \min(G(x, p), H(x, p)) = 0\},$$

then, by [32, Theorem 3.1] or [10, Corollary 3.4], there exist nonzero vectors (r, λ, μ, ξ) with $r \geq 0$ such that the following conditions hold:

(i) $0 \in r\partial_x f(x^*, p^*) + \sum_{i=1}^{m_1} \lambda_i \partial_x g_i(x^*, p^*) + \sum_{j=1}^{m_2} \partial_x(\mu_j h_j)(x^*, p^*)$ $+ \sum_{l=1}^{m} \partial_x(\xi_l \min(G_l, H_l))(x^*, p^*) + \mathcal{N}_{\mathcal{C}}(x^*)$, $\lambda \geq 0$, $\lambda_i = 0$ $(i \notin I_g^*)$;

(ii) there exists a sequence $\{x^k\} \subseteq \mathcal{C}$ converging to x^* such that, for each k,

$$\begin{aligned}
\lambda_i > 0 &\Longrightarrow \lambda_i g_i(x^k, p^*) > 0,\\
\mu_j \neq 0 &\Longrightarrow \mu_j h_j(x^k, p^*) > 0,\\
\xi_l \neq 0 &\Longrightarrow \xi_l \min(G_l(x^k, p^*), H_l(x^k, p^*)) > 0,
\end{aligned}$$

and $\{h(\cdot, p^*), g(\cdot, p^*), \min(G_l, H_l)(\cdot, p^*)\}$ are all differentiable at x^k.

We investigate $\partial_x(\xi_l \min(G_l, H_l))(x^*, p^*)$ in the following two cases.

(1) $\xi_l \geq 0$: It follows from Lemma 4.10 that

$$\begin{aligned}
\partial_x(\xi_l \min(G_l, H_l))(x^*, p^*) &= \xi_l \partial_x \min(G_l, H_l)(x^*, p^*)\\
&\subseteq \begin{cases} \xi_l \partial_x G_l(x^*, p^*), & l \in \mathcal{I}^*,\\ \xi_l \partial_x H_l(x^*, p^*), & l \in \mathcal{K}^*,\\ \xi_l \partial_x G_l(x^*, p^*) \cup \xi_l \partial_x H_l(x^*, p^*), & l \in \mathcal{J}^*. \end{cases}
\end{aligned}$$

(2) $\xi_l < 0$: It follows from Lemma 4.10 that

$$\begin{aligned}&\partial_x(\xi_l \min(G_l, H_l))(x^*, p^*)\\&\quad = \partial_x \max(\xi_l G_l, \xi_l H_l)(x^*, p^*)\\&\quad \subseteq \begin{cases} \partial_x(\xi_l G_l)(x^*, p^*), & l \in \mathcal{I}^*,\\ \partial_x(\xi_l H_l)(x^*, p^*), & l \in \mathcal{K}^*,\\ \{\partial_x(\alpha\xi_l G_l)(x^*, p^*) + \partial_x((1-\alpha)\xi_l H_l)(x^*, p^*) \mid 0 \le \alpha \le 1\}, & l \in \mathcal{J}^*.\end{cases}\end{aligned}$$

Therefore, by defining the multipliers $\{u, v\}$ using $\{\xi, \alpha\}$, the desired result follows from (i)–(ii) and (1)–(2) immediately. □

Remark 4.12. In the above theorem, all the nonsmooth functions are required to be differentiable at a given sequence in our nonsmooth enhanced optimality conditions in contrast to the required proximal subdifferentiability at a given sequence in [32, Theorem 1]. Because all the involved functions are required to be Lipschitzian around the point of interest, by Rademacher's theorem instead of the density theorem in [4, Theorem 3.1], the proximal subdifferentiability in [32, Theorem 1] can be replaced by the differentiability.

DEFINITION 4.13. *Given $r \ge 0$, we let $\mathscr{M}_C^r(x^*, p^*)$ denote the set of vectors $(\lambda, \mu, u, v, \zeta)$ at $x^* \in \mathcal{X}(p^*)$ such that*

(i) $0 \in \partial_{(x,p)}\mathscr{L}^r(x^*, p^*, \lambda, \mu, u, v) - (0, \zeta) + \mathcal{N}_{\mathcal{C}}(x^*) \times \{0\}$;

(ii) $\lambda \ge 0$, $\lambda_i = 0$ $(i \notin I_g^*)$, $u_i = 0$ $(i \in \mathcal{K}^*)$, $v_i = 0$ $(i \in \mathcal{I}^*)$, $u_i v_i \ge 0$ $(i \in \mathcal{J}^*)$;

(iii) *there exists a sequence $\{(x^k, p^k)\} \subseteq \mathcal{C} \times \Re^{n_2}$ converging to (x^*, p^*) such that, for each k,*

$$\begin{aligned}\lambda_i > 0 &\Longrightarrow \lambda_i g_i(x^k, p^k) > 0,\\ \mu_j \ne 0 &\Longrightarrow \mu_j h_j(x^k, p^k) > 0,\\ u_l \ne 0 &\Longrightarrow u_l \min(G_l(x^k, p^k), H_l(x^k, p^k)) > 0,\\ v_l \ne 0 &\Longrightarrow v_l \min(G_l(x^k, p^k), H_l(x^k, p^k)) > 0,\end{aligned}$$

and $\{h, g, \min(G_l, H_l)\}$ are all differentiable at (x^k, p^k).

DEFINITION 4.14. *We say that the MPEC C-quasi-normality holds at (p^*, x^*) for the region* gph $\mathcal{X}$ *if $(\lambda, \mu, u, v, 0) \in \mathscr{M}_C^0(x^*, p^*) \Longrightarrow (\lambda, \mu, u, v) = 0$.*

It is not difficult to verify that the MPEC C-quasi-normality persists in some feasible neighborhood.

Similarly to the previous subsection, we can easily get the following results.

THEOREM 4.15. *Assume that the restricted inf-compactness holds for (MPEC$_p$) around p^*. Then the value function $\mathcal{V}(p)$ is lower semicontinuous at p^*. Suppose further that, for each $x^* \in \mathcal{O}(p^*)$, (p^*, x^*) is MPEC C-quasi-normal for the constraint region* gph $\mathcal{X}$*. Then*

$$\begin{aligned}\partial\mathcal{V}(p^*) &\subseteq \bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_C^1(x^*, p^*)\},\\ \partial^\infty\mathcal{V}(p^*) &\subseteq \bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, u, v, \zeta) \in \mathscr{M}_C^0(x^*, p^*)\}.\end{aligned}$$

COROLLARY 4.16. *Assume that the restricted inf-compactness holds for problem* (MPEC$_p$) *around p^*. Suppose that, for each $x^* \in \mathcal{O}(p^*)$, (p^*, x^*) is MPEC C-quasi-normal for the constraint region* gph $\mathcal{X}$*. If*

$$\bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_C^0(x^*, p^*)\} = \{0\},$$

then the value function $\mathcal{V}$ is Lipschitzian around p^* with

$$\emptyset \neq \partial\mathcal{V}(p^*) \subseteq \bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_C^1(x^*, p^*)\}.$$

In addition to the above assumptions, if

$$\bigcup_{x^* \in \mathcal{O}(p^*)} \{\zeta \mid (\lambda, \mu, u, v, \zeta) \in \mathscr{M}_C^0(x^*, p^*)\} = \{\zeta\},$$

then $\mathcal{V}$ is strictly differentiable at p^* and $\nabla\mathcal{V}(p^*) = \zeta$.

We now consider the special case where all the functions $\{f, g, h, G, H\}$ are differentiable. In this case, Definition 4.13(i) becomes

(i)^1 $0 \in \nabla_x \mathscr{L}^r(x^*, p^*; \lambda, \mu, u, v) + \mathcal{N}_{\mathcal{C}}(x^*)$; (i)^2 $\zeta = \nabla_p \mathscr{L}^r(x^*, p^*; \lambda, \mu, u, v)$.

We define the set of the singular and nonsingular enhanced C-multipliers as the set of vectors (λ, μ, u, v) satisfying (i)^1 and Definition 4.13(ii)–(iii), and denote them by $\mathcal{M}_C^r(x^*, p^*)$, $r = 0, 1$, respectively.

The following example shows that Theorem 4.15 is much sharper than its C-counterpart [16, Theorem 4.8].

Example 4.17. Consider the following example

$$\begin{aligned} \min_x \quad & f(x) := x_1^2 + x_2^2 \\ \text{s.t.} \quad & g(x) := x_1 + x_2 + p \geq 0, \\ & 0 \leq G(x) := x_1 + p \perp H(x) := x_2 \geq 0. \end{aligned}$$

The value function $\mathcal{V}(p) = \left\{\begin{smallmatrix} 0 & p \geq 0 \\ p^2 & p < 0 \end{smallmatrix}\right.$ is a smooth function. For $p^* = 0$, the unique optimal solution $x^* = (0, 0)$. By solving the following singular and nonsingular C-stationarity systems for the parametric MPEC at $x^* \in \mathcal{X}(p^*)$,

$$-\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \lambda \geq 0, \ uv \geq 0,$$

we find the sets of singular and nonsingular C-multipliers

$$\mathbb{C}^i(x^*, p^*) := \{(1, 1, 1)\lambda \mid \lambda \geq 0\}, \quad i = 0, 1.$$

Since the set of singular C-multipliers contains a nonzero vector, [16, Theorem 4.8] is not applicable and one cannot even get the Lipschitz continuity of the value function. However, for any sequence $(x^k, p^k) \to (x^*, p^*)$ and any multiplier $\lambda(1, 1, 1)$ with $\lambda > 0$, the following system of inequalities does not hold:

$$\lambda(x_1^k + x_2^k + p^k) < 0, \ \lambda \min(x_1^k + p^k, x_2^k) > 0. \tag{4.8}$$

Thus, the sets of singular and nonsingular enhanced C-multipliers are $\mathcal{M}_C^i(x^*, p^*) = \{0\}$ for $i = 0, 1$, which are contained strictly in $\mathbb{C}^i(x^*, p^*)$ for $i = 0, 1$, respectively. Therefore, the MPEC C-quasi-normality holds at $x^* \in \mathcal{X}(p^*)$. Since

$$\{\lambda + u \mid (\lambda, u, v) \in \mathcal{M}_C^0(x^*, p^*)\} = \{\lambda + u \mid (\lambda, u, v) \in \mathcal{M}_C^1(x^*, p^*)\} = \{0\},$$

by Corollary 4.16, we get that the value function is strictly differentiable at p^* with $\nabla\mathcal{V}(p^*) = 0$.

Acknowledgments. The authors are grateful to the anonymous referees for their helpful comments and suggestions.

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